Rigidity of Formations with Additional Subtended-angle Constraints

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December 7, 2017
ANU Workshop on Systems and Control, Canberra
1 Background Knowledge

2 Problem Formulation

3 Weak Rigidity

4 On-going Research

5 Summary
Multi-agent formation

- A geometrical shape formed by multiple agents in a space
- Represented by a graph (vertices=agents)
- Examples:
  - A group of vehicles
  - A group of flying multi-copters
Constraints for a formation

- Distance constraints could define a formation.
- \( d_1 = d_2 = d_3 = 1 \): a regular triangle formation
- \( d_1 = 4, \ d_2 = 3, \ d_3 = 5 \): a right triangle formation
Constraints for a formation

- Insufficient constraints may not define a unique formation shape.
- $d_1 = d_2 = 1$: infinitely many non-congruent shapes satisfying the distance constraints
Rigidity is a condition so that the formation of interest is (at least locally) uniquely defined under given distance constraints.

(Distance) rigidity has been widely studied in the literature.¹

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Motivation: Main idea

- By the law of cosines, we have

\[(d_{ik})^2 = (d_{ij})^2 + (d_{jk})^2 - 2d_{ij}d_{jk} \cos \theta_{ik}^j.\]

- \(d_{ij}, d_{jk},\) and \(\theta_{ik}^j\) determine the unique distance between agents.
- An angle constraint can be equivalently converted to a distance constraint.
1. Background Knowledge

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Mixed constraints

- Suppose that there is an angle constraint in addition to the existing distance constraints.
- If the given constraints are satisfied, would the formation shape be unique?
Problem

Find an exact condition using a new concept of rigidity to distinguish whether the given mixed constraints can define a unique formation shape.
Background Knowledge

Problem Formulation

Weak Rigidity

On-going Research

Summary
Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of all vertices (agents) of the graph $\mathcal{G}$, and $\mathcal{E}$ is the set of all edges of the same graph.

Let $\mathbf{p}_i \in \mathbb{R}^2$ denote the position vector of agent $i$.

$\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_{|\mathcal{V}|}) \in \mathbb{R}^{2|\mathcal{V}|}$ is called a realization of $\mathcal{G}$ in $\mathbb{R}^2$.

$(\mathcal{G}, \mathbf{p})$ is called a framework (formation).

$\mathbf{p}_{ij} = \mathbf{p}_i - \mathbf{p}_j$.

**Assumption**

Realizability: There exists at least one realization satisfying the given distance and angle constraints.
Notation and terminologies

- For two realizations $p$ and $q$ of $G$, we say $p$ and $q$ are congruent if $\|p_{ij}\| = \|q_{ij}\|$ for all $i, j \in V$.
- Two frameworks $(G, p)$ and $(G, q)$ are said to be congruent if $p$ and $q$ are congruent.
- If $\|p_{ij}\| = \|q_{ij}\|$ for all $\{i, j\} \in E$, we say $(G, p)$ and $(G, q)$ are equivalent.
Distance rigidity of a framework

Definition (Asimow and Roth, 1978)

A given framework \((G, p)\) is rigid in \(\mathbb{R}^2\) if there exists a neighborhood \(B_p \subseteq \mathbb{R}^{2|V|}\) of \(p\) such that each framework \((G, q), q \in B_p\), equivalent to \((G, p)\) is congruent to \((G, p)\).

(Asimow and Roth, 1978)²

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²Asimow and Roth, “The rigidity of graphs”.
Subtended-angle constraints

- For a given graph $G = (V, E)$, consider some distance constraints $d_{ij} \in [0, +\infty)$ and subtended-angle constraints $\theta_{km} \in [0, \pi]$.
  - $d_{ij}$: distance constraint assigned to $\{i, j\} \in E$
  - $\theta_{km}$: angle constraint assigned to $\{k, l\}$ and $\{l, m\}$ in $E$

- Let $A = \left\{ (i, \{j, k\}) \mid \theta_{jk}^i \text{ is assigned to } \{i, j\}, \{i, k\} \in E \right\}$.

Assumption

1. The distance constraints are assigned to all edges in $E$.
2. The angle constraints are assigned to some neighboring edge pairs.
Subtended-angle constraints

Example:

- Distance constraints: $d_{12}, d_{14}, d_{23}, d_{34}$
- Angle constraint: $\theta_{24}^1$
- $\mathcal{A} = \{(1, \{2, 4\})\}$. 

```
  O
/   \    
O   O   O
  \   / 
   \ /  
    1 \ 2

\theta_{24}^1

G  \   \    \  
/     \   \   
O     O   O   O
  \   \   \  
   \   \   \ 
    3 \ 3 4
```
Subtended-angle constraints

Example: (Exception)

- Distance constraints: $d_{14}, d_{23}, d_{34}$
- Angle constraint: $\theta_{24}^1, \theta_{24}^3$.
- $\mathcal{A} = \{(1, \{2, 4\}), (3, \{2, 4\})\}$.

Question: Is it rigid? or not rigid?
Subtended-angle constraints

Only consider the left cases. The right cases could be defined similarly; but a more general setup is required.
Strong equivalence

Definition (strong equivalence)

Under a given angle set $\mathcal{A}$ of a graph $\mathcal{G}$ with $|\mathcal{V}| \geq 3$, two frameworks $(\mathcal{G}, \mathbf{p})$ and $(\mathcal{G}, \mathbf{q})$ are said to be \textit{strongly equivalent} if the following two conditions hold:

- $\forall \{v, w\} \in \mathcal{E}, \|\mathbf{p}_{vw}\| = \|\mathbf{q}_{vw}\|,$
- $\forall (i, \{j, k\}) \in \mathcal{A}, \angle \mathbf{p}_{jk}^i = \angle \mathbf{q}_{jk}^i.$

Notation:

- $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$ for two vectors $\mathbf{v}_i$ and $\mathbf{v}_j$.
- $\angle \mathbf{v}_{jk}^i \in [0, \pi]$: the angle subtended by $\mathbf{v}_{ji}$ and $\mathbf{v}_{ki}$ for three vectors $\mathbf{v}_i$, $\mathbf{v}_j$, and $\mathbf{v}_k$ provided that neither $\mathbf{v}_{ji}$ nor $\mathbf{v}_{ki}$ is empty.
Strong equivalence & Weak rigidity

Remark

*If two frameworks* \( (G, p) \) and \( (G, q) \) *are strongly equivalent, then they are equivalent, but the converse is not true.*

- If \( A = \emptyset \), then the definition of strong equivalence agrees to the definition of equivalence.

Definition (weak rigidity)

A given framework \( (G, p) \) with an associated angle set \( A \) is weakly rigid in \( \mathbb{R}^2 \) if there exists a neighborhood \( B_p \subseteq \mathbb{R}^{|V|} \) of \( p \) such that each framework \( (G, q) \), \( q \in B_p \), strongly equivalent to \( (G, p) \) is congruent to \( (G, p) \).

- If \( A = \emptyset \), the definition of weak rigidity coincides with the definition of classical distance rigidity.
A modified graph

Let \( \bar{\mathcal{G}}, \tilde{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}) \), be a graph modified from \( \mathcal{G}, \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \), in such a way that:

- \( \bar{\mathcal{V}} = \mathcal{V} \),
- \( \bar{\mathcal{E}} = \{\{i, j\} \mid \{i, j\} \in \mathcal{E} \lor \exists k \in \mathcal{V} \text{ s.t. } (k, \{i, j\}) \in \mathcal{A}\} \),
- \( \bar{\mathcal{A}} = \emptyset \).
A modified graph

Example:

- $\mathcal{V} = \{1, 2, 3, 4\}$, $\mathcal{E} = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$,
  $\mathcal{A} = \{(1, \{2, 4\})\}$.

- $\mathcal{\bar{V}} = \{1, 2, 3, 4\}$, $\mathcal{\bar{E}} = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{2, 4\}\}$, $\mathcal{\bar{A}} = \emptyset$. 

\[
\begin{align*}
\theta_{24}^1
\end{align*}
\]
Main result

Theorem

A framework \((\mathcal{G}, \mathbf{p})\) with an angle set \(\mathcal{A}\) is weakly rigid in \(\mathbb{R}^2\) if and only if \((\bar{\mathcal{G}}, \mathbf{p})\) is rigid in \(\mathbb{R}^2\).

Proof: (Sufficient condition) Suppose that \((\bar{\mathcal{G}}, \mathbf{p})\) is rigid in \(\mathbb{R}^2\). Then, there exists a neighborhood \(B_p \subseteq \mathbb{R}^{2|\mathcal{V}|}\) of \(\mathbf{p}\) such that for any \(\mathbf{q} \in B_p\), if \((\bar{\mathcal{G}}, \mathbf{p})\) and \((\bar{\mathcal{G}}, \mathbf{q})\) are equivalent, then \(\mathbf{p}\) and \(\mathbf{q}\) are congruent. Now consider \((\mathcal{G}, \mathbf{q})\) with \(\mathbf{q} \in B_p\) such that \((\mathcal{G}, \mathbf{p})\) and \((\mathcal{G}, \mathbf{q})\) are strongly equivalent. Then, it is true that

\[
\forall \{v, w\} \in \mathcal{E}, \quad \|p_{vw}\| = \|q_{vw}\|, \quad (1)
\]

\[
\forall (i, \{j, k\}) \in \mathcal{A}, \quad \angle p_{jk}^i = \angle q_{jk}^i. \quad (2)
\]

Then, we can state that

\[
\forall (i, \{j, k\}) \in \mathcal{A}, \quad \|p_{jk}\| = \|q_{jk}\|. \quad (3)
\]
Main result

because we have

\[ \|p_{jk}\|^2 = \|p_{ij}\|^2 + \|p_{ik}\|^2 - 2\|p_{ij}\|\|p_{ik}\| \cos \angle p_{jk}^i \]

\[ = \|q_{ij}\|^2 + \|q_{ik}\|^2 - 2\|q_{ij}\|\|q_{ik}\| \cos \angle q_{jk}^i \]

\[ = \|q_{jk}\|^2, \]

from (1) and (2). Therefore, we have \(\|p_{ij}\| = \|q_{ij}\|\) for all \(\{i,j\} \in \bar{E}\) from (1) and (3), which means that \((\bar{G}, p)\) and \((\bar{G}, q)\) are equivalent, and \(p\) and \(q\) are congruent from rigidity of \((\bar{G}, p)\). Consequently, we have shown that there exists a neighborhood \(B_p\) in which \(p\) and \(q\) are congruent under strong equivalence of \((G, p)\) and \((G, q)\), which means that \((G, p)\) is weakly rigid in \(\mathbb{R}^2\) (by the definition of weakly rigidity).
(Necessary condition) Suppose that $(G, p)$ is weakly rigid in $\mathbb{R}^2$. Then, there exists a neighborhood $B_p \subseteq \mathbb{R}^{2|\mathcal{V}|}$ of $p$ such that for each $q \in B_p$, if $(G, p)$ and $(G, q)$ are strongly equivalent, then $p$ and $q$ are congruent. Consider an arbitrary $q \in B_p$ such that $(\bar{G}, p)$ and $(\bar{G}, q)$ are equivalent. Thus, we have

$$\forall \{i, j\} \in \bar{E}, \|p_{ij}\| = \|q_{ij}\|.$$  \hspace{1cm} (4)

Then, it is true that

$$\forall (i, \{j, k\}) \in A, \quad \angle p^i_{jk} = \angle q^i_{jk},$$ \hspace{1cm} (5)

because we have

$$\angle p^i_{jk} = \arccos \left[ \frac{\|p_{ij}\|^2 + \|p_{ik}\|^2 - \|p_{jk}\|^2}{2\|p_{ij}\|\|p_{ik}\|} \right]$$

$$= \arccos \left[ \frac{\|q_{ij}\|^2 + \|q_{ik}\|^2 - \|q_{jk}\|^2}{2\|q_{ij}\|\|q_{ik}\|} \right].$$
\[ \angle q^{i}_{jk}, \]

from (4). Moreover, we know that

\[ \forall \{i,j\} \in \mathcal{E}, \|p_{ij}\| = \|q_{ij}\|, \quad (6) \]

from that \( \mathcal{E} \subseteq \bar{\mathcal{E}} \). Since \((\bar{\mathcal{G}}, p)\) is weakly rigid, \(p\) and \(q\) must be congruent due to (5) and (6) under equivalence of \((\bar{\mathcal{G}}, p)\) and \((\bar{\mathcal{G}}, q)\), which means that \((\bar{\mathcal{G}}, p)\) is rigid in \(\mathbb{R}^2\).
Example 1

- \((\bar{G}, p)\) is not rigid.
- \((G, p)\) is not weakly rigid.
Example II

- $(\bar{G}, p)$ is rigid.
- $(G, p)$ is weakly rigid.

(a) $(G, p)$

(b) $(\bar{G}, p)$
Example:

Figure. (a) An example of non-rigid but weakly rigid framework.

Figure. (b) An example of rigid framework.
Weak rigidity matrix (Rigidity matrix) - Credit also goes to Brian

Generic weak rigidity (Generic weak rigidity vs. weak rigidity vs. Infinitesimally weak rigidity)\(^3\):
- \( p_{ij} = p_i - p_j, \forall (i, j) \in E. \)
- \( \theta^k_{ij} = (i, \{j, k\}), \forall (i, \{j, k\}) \in A; \cos \theta^k_{ij} = \left[ \frac{\|p_{ik}\|^2 + \|p_{jk}\|^2 - \|p_{ij}\|^2}{2\|p_{ik}\|\|p_{jk}\|} \right]. \)
- \( e_g \equiv p_{ij}, \forall g \in \{1, ..., m_e\}. \)
- \( A_h \equiv \cos \theta^k_{ij}, \forall h \in \{1, ..., m_a\}. \)

The weak rigidity function is defined:
\( F_w(p) \equiv [\|e_1\|^2, ..., \|e_{m_e}\|^2, A_1, ..., A_{m_a}]^T \in \mathbb{R}^{2(m_e + m_a)}. \)

The weak rigidity matrix is defined as the Jacobian of the weak rigidity function: \( R_w(p) \equiv \frac{\partial F_w(p)}{\partial p} \in \mathbb{R}^{2(m_e + m_a) \times 2|V|}. \)

\(^3\)Topological concept - Given a realization, it is a weak rigid; then almost all realizations will be weak rigid. But it is not clear since we consider angles. Infinitesimally weak rigidity \( \Rightarrow \) Weak rigidity
Let \((G, p)\) denote a framework in \(\mathbb{R}^2\) with \(|V| \geq 3\). Suppose that \(p\) is generic and \(m_e \geq 1\). Then, \((G, p)\) is infinitesimally weak rigid in \(\mathbb{R}^2\) if and only if the weak rigidity matrix \(R_w(p)\) has rank \(2|V| - 3\) (**Issue:** upto partial scaling or entire scaling?).

Let \((G, p)\) denote a framework in \(\mathbb{R}^2\) with \(|V| \geq 3\). Suppose that \(p\) is generic and \(m_e = 0\). Then, \((G, p)\) is infinitesimally weak rigid in \(\mathbb{R}^2\) if and only if the weak rigidity matrix \(R_w(p)\) has rank \(2|V| - 4\) (upto translations, rotations, and (partial or entire) scalings).\(^a\)

Weak rigidity matrix (Rigidity matrix) - Credit also goes to Brian

Example:
The following two examples have the same rank (= 5).

(a) An example of rigid framework. 

(b) An example of non-rigid but weakly rigid framework.
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On-going Research

- How to extend the results to 3-dimensional space?
- Extension to bearing-based cases
- Extension to a large-size graph starting from $K_3$ by Henneberg extensions (0- and 1-extensions)
- One-to-one mapping between distance-constraints and angle-constraints
Henneberg extensions: 0-extension
Henneberg extensions: 1-extension
Extension to the 3-Dimensional Space

Conjecture

A framework \((G, p)\) with an angle set \(A\) is weakly rigid in \(\mathbb{R}^3\) if and only if \((\overline{G}, p)\) is rigid in \(\mathbb{R}^3\).

Figure. (a) An example of non-rigid but weakly rigid framework in the 3-dimensional space.

Figure. (b) An example of rigid framework in the 3-dimensional space.
Rigidity of Bearing-based Formations with Additional Subtended-angle Constraints

The following two conditions can be equivalent w.r.t. a triangular formation:

- Two bearings with another subtended angle.
- Three inter-neighbor bearings.

**Figure.** (a) An example of non-rigid but bearing-based weakly rigid framework. **Figure.** (b) An example of bearing-based rigid framework.
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Conclusion

- Use a triangle as the primal component of the analysis
- One-to-one mapping between distance-constraints and angle-constraints

Main reference:
Myoung-Chul Park, Hong-Kyong Kim, Hyo-Sung Ahn, “Rigidity of Distance-based Formations with Additional Subtended-angle Constraints,” *Proc. of the 17th International Conference on Control, Automation and Systems*, Je-ju, Korea, October 18-21, 2017

On-going research:
- Generalization in 3-dimensional space.
- Henneberg extensions
- Extension to the bearing-based weakly rigidity.
- Formulation of the cases with “subtended angles without full adjacent edges”.
- Network localization
References I


Thank you for your attention.

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