Boolean Gossip Networks

Guodong Shi

Research School of Engineering
The Australian National University, Canberra, Australia

ANU Workshop on Systems and Control, 2017
Joint work with
Bo Li and Hongsheng Qi, Academy of Mathematics and Systems Science, China
Junfeng Wu, School of Control Automation, Zhejiang University, China
and Alexandre Proutiere, KTH Automatic Control, Sweden
• Genes are DNA and RNA that are biological codes of molecules for their functions
• Genes are DNA and RNA that are biological codes of molecules for their functions
• Genes, when turned on, are expressed into RNA and protein as functional gene products
• Genes are DNA and RNA that are biological codes of molecules for their functions
• Genes, when turned on, are expressed into RNA and protein as functional gene products
• Some genes (or proteins) can control the expressions of other genes, which are known as regulator genes
• Genes are DNA and RNA that are biological codes of molecules for their functions
• Genes, when turned on, are expressed into RNA and protein as functional gene products
• Some genes (or proteins) can control the expressions of other genes, which are known as regulator genes
• Gene regulators interact with each other forming coupled dynamical evolutions
Gene Regulatory Networks

- Genes are DNA and RNA that are biological codes of molecules for their functions
- Genes, when turned on, are expressed into RNA and protein as functional gene products
- Some genes (or proteins) can control the expressions of other genes, which are known as regulator genes
- Gene regulators interact with each other forming coupled dynamical evolutions
Gene Regulatory Networks

[Ma et al. 2014]
Kauffman’s Probabilistic Boolean Network
Kauffman’s Probabilistic Boolean Network

• In a network of n nodes, each node holds a binary state at discretized time instants
Kauffman’s Probabilistic Boolean Network

• In a network of n nodes, each node holds a binary state at discretized time instants
• There are a finite number of mappings from n-dimensional binary space to itself
Kauffman’s Probabilistic Boolean Network

- In a network of $n$ nodes, each node holds a binary state at discretized time instants
- There are a finite number of mappings from $n$-dimensional binary space to itself
- Each node randomly selects one of the mappings describing how it interacts with the remainder of the network
Kauffman’s Probabilistic Boolean Network

• In a network of $n$ nodes, each node holds a binary state at discretized time instants
• There are a finite number of mappings from $n$-dimensional binary space to itself
• Each node randomly selects one of the mappings describing how it interacts with the remainder of the network

$$x_i(t) \in \{0, 1\}$$
Kauffman’s Probabilistic Boolean Network

- In a network of n nodes, each node holds a binary state at discretized time instants
- There are a finite number of mappings from n-dimensional binary space to itself
- Each node randomly selects one of the mappings describing how it interacts with the remainder of the network

\[ x_i(t) \in \{0, 1\} \]

\[ x_i(t + 1) = f_i(x_1(t), \ldots, x_n(t)) \]
Kauffman’s Probabilistic Boolean Network

\[ x_i(t + 1) = f_i(x_1(t), \ldots, x_n(t)) \]

\[ x_i(t) \in \{0, 1\} \]

[Kauffman 1969]
Kauffman’s Probabilistic Boolean Network

\[ x_i(t + 1) = f_i(x_1(t), \ldots, x_n(t)) \]
Kauffman’s Probabilistic Boolean Network

\[ x_i(t + 1) = f_i(x_1(t), \ldots, x_n(t)) \]

Finding a singleton attractor is NP hard!
[Akutsu et al. 1998]
Kauffman’s Probabilistic Boolean Network

\[ x_i(t + 1) = f_i(x_1(t), \ldots, x_n(t)) \]

Finding a singleton attractor is NP hard!

[Akutsu et al. 1998]

[Shmulevich et al. 2002; Brun et al. 2005; Cheng and Qi 2009; Chaves and Carta 2014;...]

Kauffman’s Proposal Revisited

\[ x_i(t + 1) = f_i(x_1(t), \ldots, x_n(t)) \]
Kauffman’s Proposal Revisited

\[ x_i(t + 1) = f_i(x_1(t), \ldots, x_n(t)) \]

Locality of gene interactions: each regulator gene interacts only with a few (2 or 3) neighboring genes.

“element received just two inputs from other elements is biologically reasonable” [Kauffman 1969]
A Boolean Gossip Network Model
Pairwise Boolean Interactions

\[ V = \{1, \ldots, n\} \]

\[ G = (V, E) \]

\[ x_i(t) \in \{0, 1\} \]
Pairwise Boolean Interactions
Pairwise Boolean Interactions

\[ x_i(t) \in \{0, 1\} \]
Pairwise Boolean Interactions

\[ x_i(t) \in \{0, 1\} \]

\[ x_j(t) \in \{0, 1\} \]
Pairwise Boolean Interactions

\[ x_i(t) \in \{0, 1\} \]

\[ x_j(t) \in \{0, 1\} \]

[Karp et al. 2000, Boyd et al. 2006]
Pairwise Boolean Interactions
Pairwise Boolean Interactions

\[ x_i(t) \in \{0, 1\} \]

\[ x_j(t) \in \{0, 1\} \]

\[ H = \{ \circ_1, \ldots, \circ_9, \circ_A, \ldots, \circ_F \} \]
Pairwise Boolean Interactions

\[ x_i(t) \in \{0, 1\} \]

\[ x_j(t) \in \{0, 1\} \]

\[ H = \{ \ominus_1, \ldots, \ominus_9, \ominus_A, \ldots, \ominus_F \} \]

Admissible interaction set

\[ C = \{ \ominus_{c_1}, \ldots, \ominus_{c_q} \} \]
Boolean Gossip

\( x_i(t) \in \{0, 1\} \quad x_j(t) \in \{0, 1\} \)

\[
H = \{ \odot_1, \ldots, \odot_9, \odot_A, \ldots, \odot_F \}
\]

\[
C = \{ \odot_{c_1}, \ldots, \odot_{c_q} \}
\]
Boolean Gossip

\[ x_i(t) \in \{0, 1\} \quad x_j(t) \in \{0, 1\} \]

\[ H = \{ \circ_{01}, \ldots, \circ_9, \circ_A, \ldots, \circ_F \} \]

\[ C = \{ \circ_{c_1}, \ldots, \circ_{c_q} \} \]

\[
\begin{cases}
  x_i(t+1) = x_i(t) \circ_{c_k} x_j(t), \\
  x_j(t+1) = x_j(t) \circ_{c_l} x_i(t), \\
  x_m(t+1) = x_m(t), \quad m \notin \{i, j\}. 
\end{cases}
\]
Induced Markov Chain
Induced Markov Chain

\[ X_t = (x_1(t) \ldots x_n(t))^\top \]
Induced Markov Chain

\[ X_t = (x_1(t) \ldots x_n(t))^\top \]

\[ S_n = \{ [s_1 \ldots s_n] : s_i \in \{0, 1\}, i \in V \} \]
Induced Markov Chain

\[ X_t = (x_1(t) \ldots x_n(t))^\top \]

\[ S_n = \{[s_1 \ldots s_n] : s_i \in \{0, 1\}, i \in V\} \]

\[ P = [P_{s_1 \ldots s_n}[q_1 \ldots q_n]] \in \mathbb{R}^{2^n \times 2^n} : \\
\]

\[ P_{s_1 \ldots s_n}[q_1 \ldots q_n] := \mathbb{P}\left(X_{t+1} = [q_1 \ldots q_n] \bigg| X_t = [s_1 \ldots s_n]\right). \]
Induced Markov Chain

\[ X_t = (x_1(t) \ldots x_n(t))^\top \]

\[ S_n = \{[s_1 \ldots s_n] : s_i \in \{0, 1\}, i \in V\} \]

\[ P = [P_{s_1 \ldots s_n[q_1 \ldots q_n]}] \in \mathbb{R}^{2^n \times 2^n} : \]

\[ P_{[s_1 \ldots s_n][q_1 \ldots q_n]} := \mathbb{P}\left(X_{t+1} = [q_1 \ldots q_n] \mid X_t = [s_1 \ldots s_n]\right).\]
Positive Boolean Interactions
Positive Boolean Operations

\( \land \quad \lor \quad \neg \)
Positive Boolean Operations

\[ \wedge \quad \vee \quad \neg \]

\[ C_{pst} = \{ \vee, \wedge \} \]
Proposition.
There exists a Bernoulli random variable $x_\ast$ such that

$$\mathbb{P}\left( \lim_{t \to \infty} x_i(t) = x_\ast, \text{ for all } i \in V \right) = 1.$$ 

The limit $x_\ast$ satisfies

$$\mathbb{E}\{x_\ast\} = \left[ (I_{2n-2} - Q)^{-1} R \right]_{X_0[1...1]}.$$
Mean-field Approximation for Regular Graphs

\[ \delta(t) = \sum_{i=1}^{n} \frac{x_i(t)}{n} \]
Mean-field Approximation for Regular Graphs

\[ \delta(t) = \sum_{i=1}^{n} x_i(t)/n \]

\[ \frac{d}{ds} \delta(s) = p_{\star}^2 \cdot 2\delta(s)(1 - \delta(s)) - (1 - p_{\star})^2 \cdot 2\delta(s)(1 - \delta(s)) \]
Mean-field Approximation for Regular Graphs

\[ \delta(t) = \sum_{i=1}^{n} x_i(t)/n \]

\[ \frac{d}{ds} \delta(s) = p_\star^2 \cdot 2\delta(s)(1 - \delta(s)) - (1 - p_\star)^2 \cdot 2\delta(s)(1 - \delta(s)) \]

\[ \delta(s) = \frac{\delta(0)}{(1 - \delta(0))e^{2(1-2p_\star)s} + \delta(0)}. \]
Numerical Example

Node Proportion with State 1

Simulated Realization $p_*=0.49$
ODE Approximation $p_*=0.49$
Simulated Realization $p_*=0.51$
ODE Approximation $p_*=0.51$
Communication Classes

- In a Markov chain, two states in the state space communicate with each other if they are accessible from each other.
Communication Classes

- In a Markov chain, two states in the state space communicate with each other if they are accessible from each other.
- Communication relationship forms an equivalence relation over the state space; the resulting equivalence classes are called communication classes.
Communication Classes

- In a Markov chain, two states in the state space communicate with each other if they are accessible from each other.
- Communication relationship forms an equivalence relation over the state space; the resulting equivalence classes are called communication classes.

\[ M_G(C) = (S_n, P) \]
Communication Classes

- In a Markov chain, two states in the state space communicate with each other if they are accessible from each other.
- Communication relationship forms an equivalence relation over the state space; the resulting equivalence classes are called communication classes.

\[ M_G(C) = (S_n, P) \]  \[ \chi_C(G) \]
Communication Classes

**Theorem**
There hold

(i) $\chi_{c_{pst}}(G) = 2n$ if $G$ is a line graph;

(ii) $\chi_{c_{pst}}(G) = m + 3$ if $G$ is a cycle graph with $n = 2m$; $\chi_{c_{pst}}(G) = m + 2$ if $G$ is a cycle graph with $n = 2m + 1$;

(iii) $\chi_{c_{pst}}(G) = 5$ if $G$ is neither a line nor a cycle, and contains no odd cycle;

(iv) $\chi_{c_{pst}}(G) = 3$ if $G$ is not a cycle graph but contains an odd cycle.
Communication Classes

\[ \chi_{c_{\text{pst}}} (G) = 2n \]
Communication Classes
Communication Classes

\[ \chi_{c_{pst}}(G) = 3 \]
General Boolean Interactions
General Boolean Operations

\[ H = \{ \circ_1, \ldots, \circ_9, \circ_A, \ldots, \circ_F \} \]

\[ C = \{ \circ_{c_1}, \ldots, \circ_{c_q} \} \]
General Boolean Operations

\[ H = \{ \odot_1, \ldots, \odot_9, \odot_A, \ldots, \odot_F \} \]

\[ C = \{ \odot c_1, \ldots, \odot c_q \} \]

\[ 2^{16} - 1 = 65535 \]
General Boolean Operations

\[ H = \{ \circ_1, \ldots, \circ_9, \circ_A, \ldots, \circ_F \} \]

\[ C = \{ \circ_{c_1}, \ldots, \circ_{c_q} \} \]

\[ 2^{16} - 1 = 65535 \]

\[ B := B_1 \cup B_2 \]

\[ B_1 = \{ C \neq \{ \circ_A \} \in 2^H : \{ \circ_A \} \subset C \subset \{ \circ_2, \circ_3, \circ_A, \circ_B \} \} \]

\[ B_2 = \{ C \in 2^H : \{ \circ_2, \circ_B \} \subset C \subset \{ \circ_2, \circ_3, \circ_A, \circ_B \} \} \]
General Boolean Operations

\[ H = \{ \circ_1, \ldots, \circ_9, \circ_A, \ldots, \circ_F \} \]

\[ C = \{ \circ c_1, \ldots, \circ c_q \} \]

\[ 2^{16} - 1 = 65535 \]

\[ \mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2 \]

\[ \mathcal{B}_1 = \{ C \neq \{ \circ_A \} \in 2^H : \{ \circ_A \} \subset C \subset \{ \circ_2, \circ_3, \circ_A, \circ_B \} \} \]

\[ \mathcal{B}_2 = \{ C \in 2^H : \{ \circ_2, \circ_B \} \subset C \subset \{ \circ_2, \circ_3, \circ_A, \circ_B \} \} \]
Communication Classes

\[ \chi_c(G) \]
Communication Classes

\[ \chi_c(G) \quad C = \{ \diamond C_1, \ldots, \diamond C_q \} \]
Communication Classes

\[ \chi_c(G) \]

\[ C = \{ \otimes_{c_1}, \ldots, \otimes_{c_q} \} \]
Absorbing Chain

Theorem

Suppose $C \in \mathcal{B}$. Then $\mathcal{M}_G(C)$ is an absorbing Markov chain if and only if $G$ does not contain an odd cycle.
Absorbing Chain

Theorem

Suppose $C \in 2^{H} \setminus \mathcal{B}$. Then $\mathcal{M}_{G}(C)$ is an absorbing Markov chain if and only if one of the following two conditions holds

(i) $C \subseteq \{0, 1, 2, 3, 4, 5, 6, 7\}$;

(ii) $C \subseteq \{1, 3, 5, 7, 9, B, D, F\}$. 
Absorbing Chain

Theorem
Suppose $C \in 2^H \setminus \mathcal{B}$. Then $\mathcal{M}_G(C)$ is an absorbing Markov chain if and only if one of the following two conditions holds

(i) $C \subseteq \{ \odot_0, \odot_1, \odot_2, \odot_3, \odot_4, \odot_5, \odot_6, \odot_7 \}$;

(ii) $C \subseteq \{ \odot_1, \odot_3, \odot_5, \odot_7, \odot_9, \odot_B, \odot_D, \odot_F \}$.

Thank you!