Australian National University

WORKSHOP ON SYSTEMS AND CONTROL

Canberra, AU
December 7, 2017

Australian National University

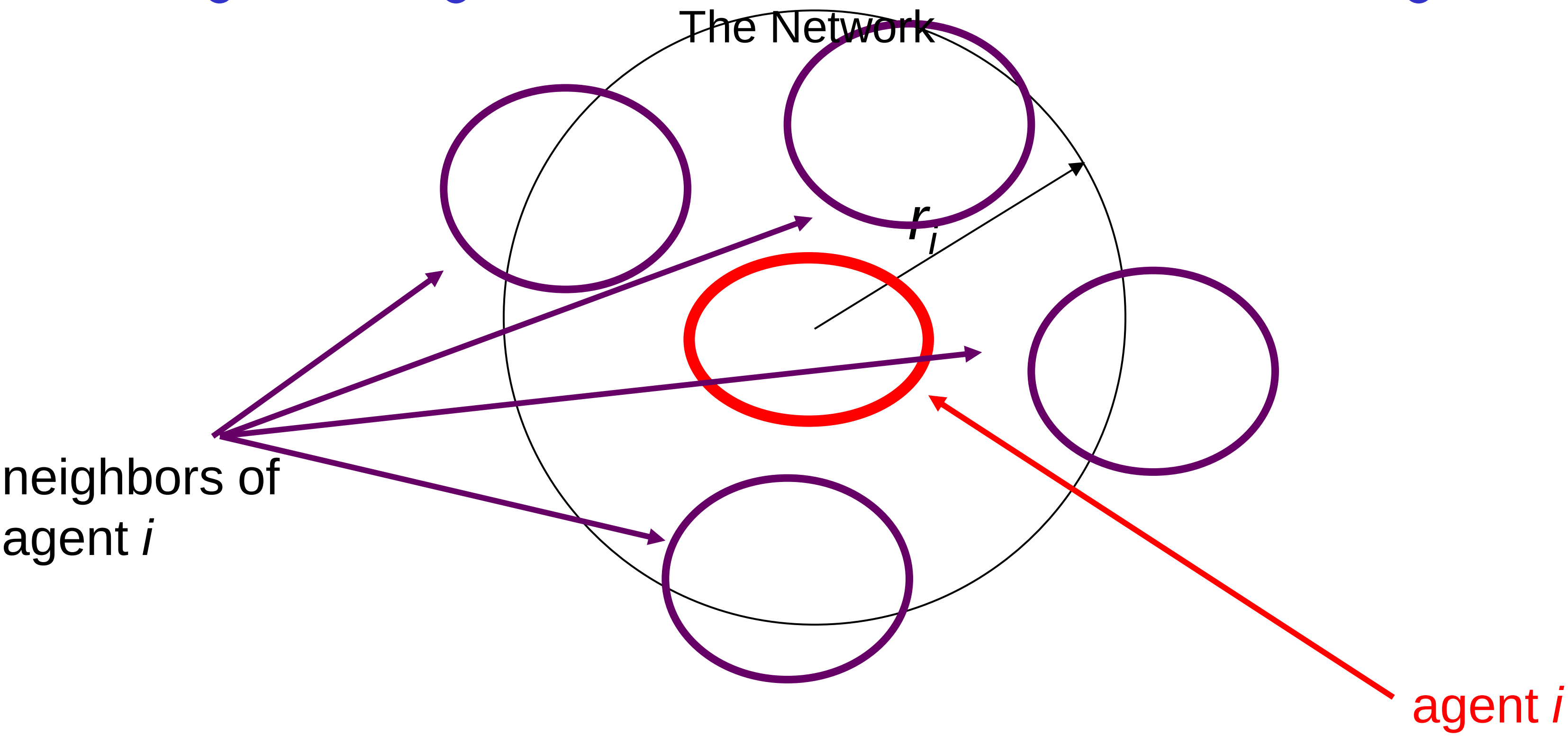
WORKSHOP ON SYSTEMS AND CONTROL

A Distributed Algorithm for Finding a Common Fixed Point of a Family of Paracontractions and Some of Its Applications

A S Morse
Yale University

Lili Wang Daniel Fulmer Ji Liu

Canberra, AU
December 7, 2017



Each agent has its own reception radius r_i

Each agent is a neighbor of itself

$m > 1$ agents with labels 1,2,.... m

Motivating Problem: Solving $Ax = b$ .... over a network of m agents

Each agent i knows a pair of real matrices $(A_i^{n_i \times n}, b_i^{n_i \times 1})$

Standing assumption: At least one solution to the equation $Ax = b$ exists where

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

There are no other assumptions about the A_i and b_i

Time is discrete: $t \in \{1,2,...\}$

Set of labels of agent i's neighbors at time t

At time t each agent

stores an $n \times 1$ state vector $x_i(t)$ which it can update.

receives the state vectors $x_j(t)$, $j \in \mathcal{N}_i(t)$, of each of its current neighbors

knows nothing more

Problem: Devise local update rules for the x_i which will cause all m agents to iteratively arrive at the same solution to $Ax = b$

Motivating Problem

Algorithm

Positive definite gain matrix chosen small enough so that

spectrum (I ¡ $A_i^0 G_i A_i$) ½ (¡ 1; 1]

$$M_i(x) = x - A_i' G_i (A_i x - b_i)$$

$$x_i(t+1) = M_i \left(\frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} x_j(t) \right), \quad t \geq 1, \qquad i \in \{1, 2, \ldots, m\}$$

Number of labels in $\mathcal{N}_i(t)$

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$Ax = b \quad \Longleftrightarrow \quad A_i x = b_i, \quad i \in \{1, 2, \ldots, m\}$$

$$A_i x = b_i \quad \Longleftrightarrow \quad M_i(x) = x$$

$$Ax = b \quad \Longleftrightarrow \quad M_i(x) = x, \quad i \in \{1, 2, \ldots, m\}$$

Therefore any solution to $Ax = b$ is a common fixed point of the M_i and conversely.

Can a distributed algorithm be constructed for computing a common fixed point for other types of possibly nonlinear M_i?

Problem Generalization

Each agent i knows a (possibly nonlinear) map

$$M_i : \mathbb{R}^n \to \mathbb{R}^n$$

Suppose the M_i have at least one common fixed point y; that is

$$M_i(y) = y, \quad i \in \{1, 2, \dots, m\}$$

Each agent i generates a state $x_i(t) \in \mathbb{R}^n$ which is its current estimate of y.

Each agent i receives the state $x_j(t)$ of each of its current neighbors $j \in \mathcal{N}_i(t)$.

Problem: Devise a distributed recursive algorithm which enables all m agents to asymptotically compute the same common fixed point of the M_i.

But for what kinds of M_i?

$$M_i(x) = x - A_i' G_i (A_i x - b_i) \qquad \sigma(I - A_i' G_i A_i) \subset (-1, 1]$$

Each M_i is a "paracontraction" with respect to the 2-norm

Paracontraction

$M : \mathbb{R}^n \to \mathbb{R}^n$ is a *paracontraction* with respect to a given norm $||\cdot||$ on $\mathbb{R}^n$ if it is continuous and

$$||M(x) - y|| < ||x - y||$$

for all $y \in \mathbb{R}^n$ satisfying $M(y) = y$ and all $x \in \mathbb{R}^n$ satisfying $M(x) \neq x$

Thus if M is a paracontraction and x is not a fixed point, then $M(x)$ is "closer" to the set of fixed points

$$\mathcal{F}(M) = \{z \ : \ M(z) = z\}$$

than x was.

Paracontraction w/r $||\cdot||_2$

Example

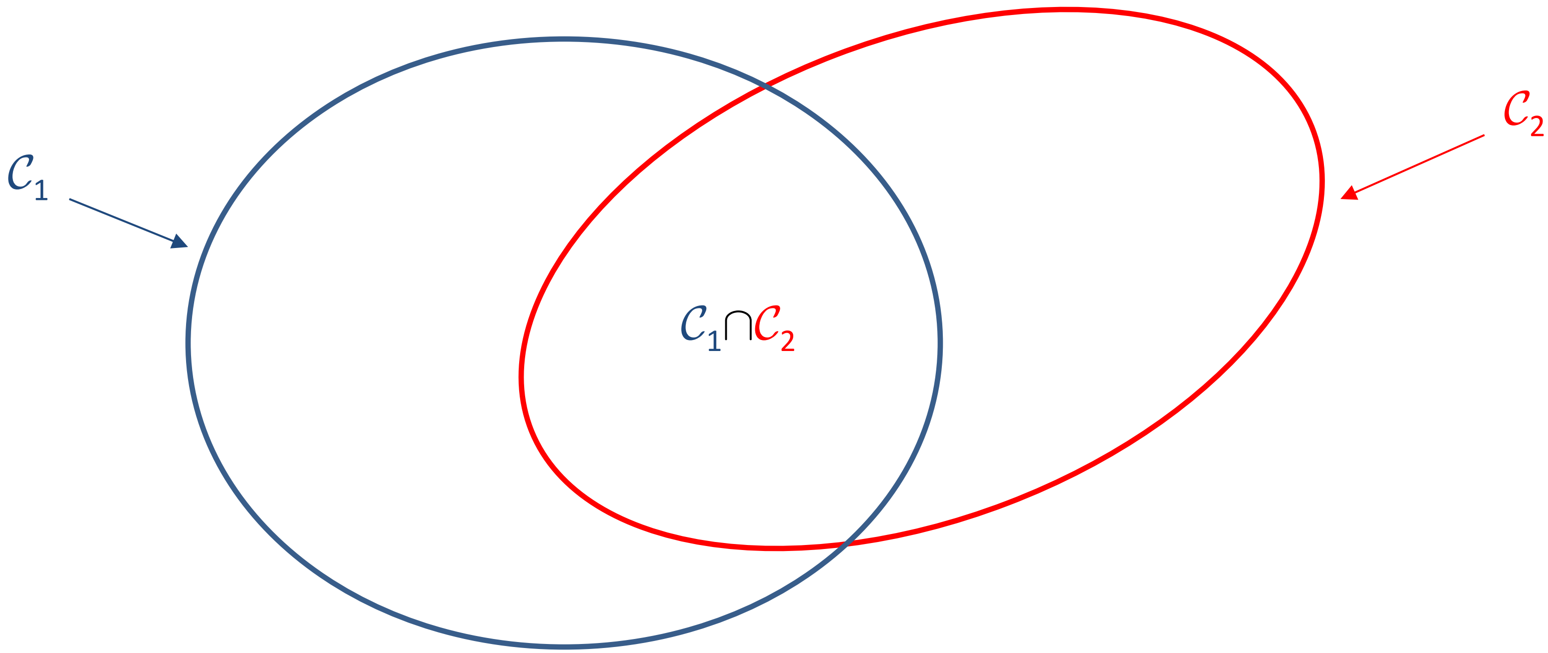
Orthogonal Projection

$$M(x) = \arg\min_{y \in \mathcal{C}} ||x - y||_2$$

$\mathcal{C}$ = closed convex set

Fixed points of M are vectors in $\mathcal{C}$

Can be used to find a vector in the intersection of *m* closed convex sets.



The Problem {again}

Paracontractions: $M_i : \mathbb{R}^n \to \mathbb{R}^n, \quad i \in \{1, 2, \ldots, m\}$

Common fixed point y: $M_i(y) = y, \quad i \in \{1, 2, \ldots, m\}$

Agent i has a state $x_i(t)$ which is to be its current estimate of a common fixed point.

Agent i knows M_i and receives the state $x_j(t)$ of each of its current neighbors $j \in \mathcal{N}_i(t)$.

Problem: Devise a distributed recursive algorithm which enables all m agents to asymptotically compute the same common fixed point of the M_i.

The Algorithm

number of labels in $\mathcal{N}_i(t)$

$$x_i(t+1) = M_i\left(\frac{1}{m_i(t)}\sum_{j\in\mathcal{N}_i(t)} x_j(t)\right), \quad t\geq 1, \qquad i\in\{1,2,\ldots,m\}$$

labels of agent *i*'s neighbors

Generalization:

$$x_i(t+1) = M_i\left(\sum_{j\in\mathcal{N}_i(t)} w_{ij}(t)x_j(t)\right), \quad t\geq 1, \qquad i\in\{1,2,\ldots,m\}$$

$$w_{ij}(t)\geq 0 \qquad\qquad \sum_{j\in\mathcal{N}_i(t)} w_{ij}(t) = 1$$

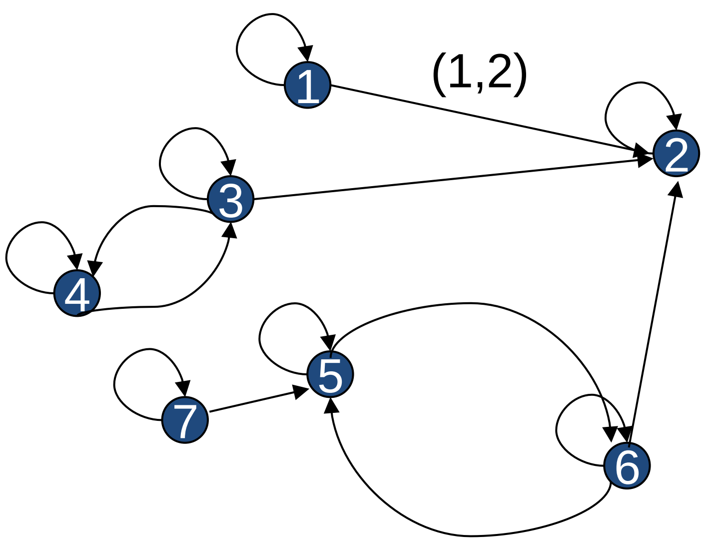
Require that for each *i* and *j*, and all $t \geq 1$, $w_{ij}(t) \in \mathcal{W}_{ij}$ = finite set

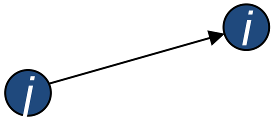
Neighbor Graph

$$x_i(t+1) = M_i\left(\frac{1}{m_i(t)}\sum_{j\in\mathcal{N}_i(t)} x_j(t)\right), \quad t \geq 1, \qquad i \in \{1,2,\ldots,m\}$$

$\mathcal{G}$ = all directed graphs with vertex set $\mathcal{V} = \{1,2,\ldots,m\}$

$\mathbb{N}(t)$ = graph in $\mathcal{G}$ with an arc from *j to i* whenever $j \in \mathcal{N}_i(t)$, $i \in \{1,2,\ldots,m\}$





j is a neighbor of *i*

Main Result

$$x_i(t+1) = M_i\left(\frac{1}{m_i(t)}\sum_{j\in\mathcal{N}_i(t)} x_j(t)\right), \quad t \geq 1, \qquad i \in \{1, 2, \ldots, m\}$$

Suppose that for some finite integer $p > 1$, the maps M_1 ,M_2, ..., M_m are all paracontractions with respect to the same norm $||\cdot||_p$ and, in addition, that the M_i all share at least one common fixed point. Then for any sequence of strongly connected neighbor graph $\mathbb{N}(t)$, $t \geq 1$, all $x_i(t)$ converge to the same point as $t \to \infty$, and this point is a fixed point of all of the M_i.

{NOLCOS 2016}

This result also holds if the sequence of neighbor graphs $\mathbb{N}(t)$, $t \geq 1$, is "repeatedly jointly strongly connected."

Repeatedly jointly strongly connected means that for some finite integer $T > 0$, the union of each successive sub-sequence of T graphs in the sequence $\mathbb{N}(t)$, $t \geq 1$, is strongly connected.

This result also holds if the agents do not share a common clock and updates are performed asynchronously. {CDC 2016}

Elsner, Koltracht, Neumann (1992):

Suppose $\mathcal{P}$ is a finite set of paracontractions $P : \mathbb{R}^n \to \mathbb{R}^n$ with respect to some norm on $\mathbb{R}^n$. Suppose the paracontractions share a common fixed point. Then the state of the iteration

$$z(t+1) = P_t(z(t)),\ t \geq 1$$

converges to a common fixed point of the paracontractions which occur in the sequence P_t, $t \geq 1$, infinitely often.

Simulation

Positive definite gain matrix chosen small enough so that

spectrum $(I$ ¡ $A_i^0 G_i A_i)$ ½ $($¡ $1; 1]$

$$M_i(x) = x - A_i' G_i (A_i x - b_i)$$

$$x_i(t+1) = M_i\left(\frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} x_j(t)\right), \quad t \geq 1, \qquad i \in \{1, 2, \ldots, m\}$$

Assume for simplicity that each A_i has linearly independent rows

Simulation

$$M_i(x) = x - A_i' G_i(A_i x - b_i)$$

$$x_i(t+1) = M_i\left(\frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} x_j(t)\right), \quad t \geq 1, \qquad i \in \{1, 2, \dots, m\}$$

Assume for simplicity that each
A_i has linearly independent rows

Simulation

$$M_i(x) = x - A_i' G_i (A_i x - b_i) \qquad (A_i A_i')^{-1} \leq G_i < 2(A_i A_i')^{-1}$$

$$x_i(t+1) = M_i \left(\frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} x_j(t) \right), \quad t \geq 1, \qquad i \in \{1, 2, \ldots, m\}$$

Assume for simplicity that each
A_i has linearly independent rows

$$y - x = 1 \qquad A_1 = [\,-1 \quad 1\,] \qquad b_1 = 1$$
$$y + 3x = 3 \qquad A_2 = [\,3 \quad 1\,] \qquad b_2 = 3$$

$$M_i(x) = x - A_i' G_i (A_i x - b_i) \qquad (A_i A_i')^{-1} \leq G_i < 2(A_i A_i')^{-1}$$

$$x_i(t+1) = M_i\left(\frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} x_j(t)\right), \quad t \geq 1, \qquad i \in \{1, 2, \ldots, m\}$$

$$G_i = (A_i A_i')^{-1}, \quad i = 1, 2 \qquad G_1 = \frac{2}{3} I \qquad G_2 = \frac{1}{6} I$$

Two simulations: one with

the other with

$$y - x = 1$$
$$y + 3x = 3$$

$$A_1 = \begin{bmatrix} -1 & 1 \end{bmatrix} \qquad b_1 = 1$$
$$A_2 = \begin{bmatrix} 3 & 1 \end{bmatrix} \qquad b_2 = 3$$

$$M_i(x) = x - A_i' G_i (A_i x - b_i) \qquad (A_i A_i')^{-1} \leq G_i < 2(A_i A_i')^{-1}$$

$$x_i(t+1) = M_i\left(\frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} x_j(t)\right), \quad t \geq 1, \qquad i \in \{1, 2, \ldots, m\}$$

$$G_i = (A_i A_i')^{-1}, \quad i = 1, 2 \qquad G_1 = \frac{2}{3} I \qquad G_2 = \frac{1}{6} I$$

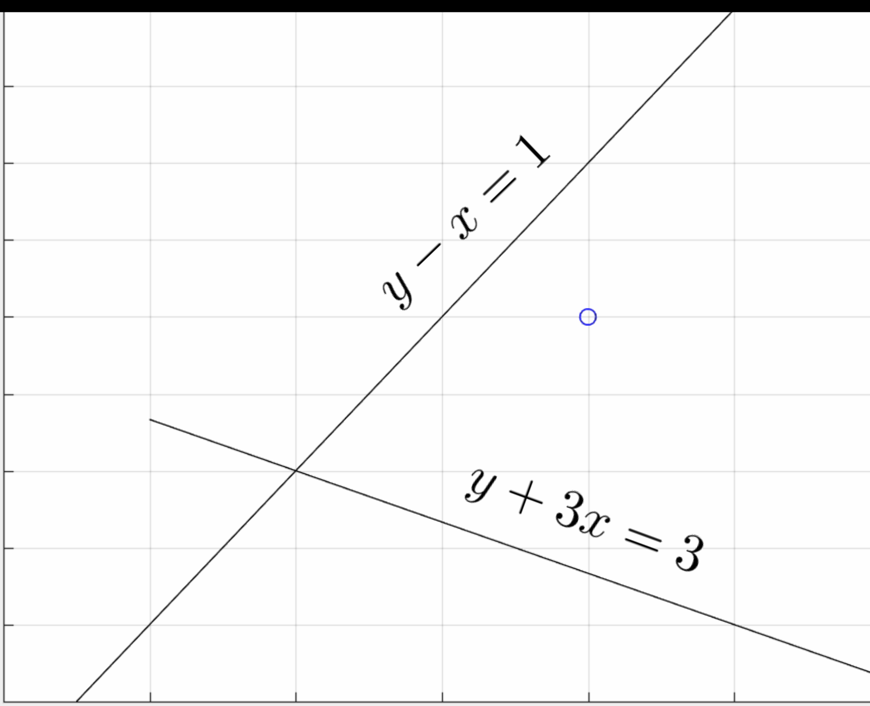
$$y - x = 1 \qquad A_1 = \begin{bmatrix} -1 & 1 \end{bmatrix} \qquad b_1 = 1$$

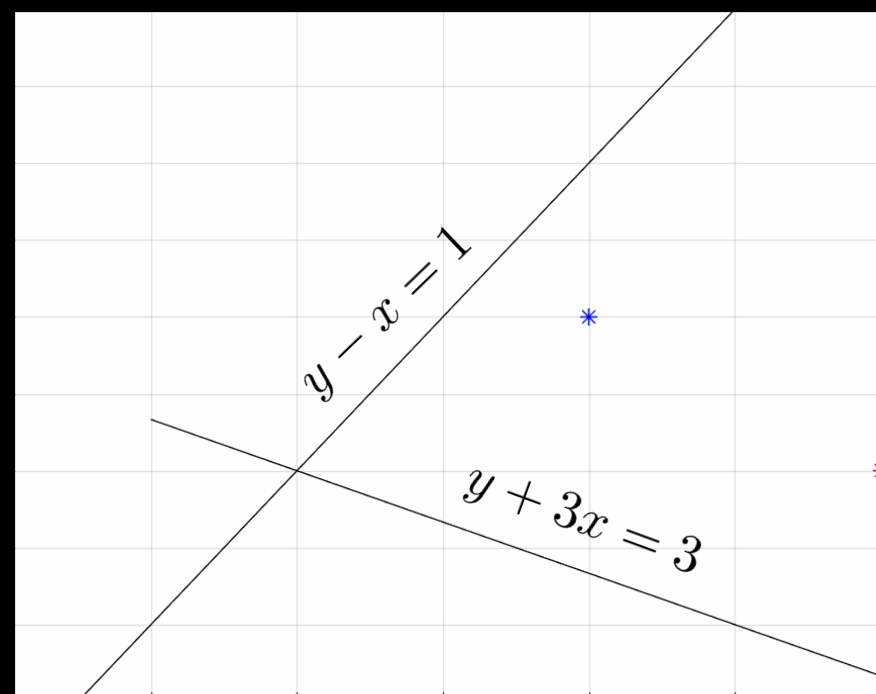
$$y + 3x = 3 \qquad A_2 = \begin{bmatrix} 3 & 1 \end{bmatrix} \qquad b_2 = 3$$

$$M_i(x) = x - A_i' G_i (A_i x - b_i) \qquad (A_i A_i')^{-1} \leq G_i < 2(A_i A_i')^{-1}$$

$$x_i(t+1) = M_i \left(\frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} x_j(t) \right), \quad t \geq 1, \qquad i \in \{1, 2, \ldots, m\}$$

$$G_i = (A_i A_i')^{-1}, \quad i = 1, 2 \qquad G_1 = \frac{2}{3} I \qquad G_2 = \frac{1}{6} I$$





Application: Distributed State Estimation

Process

n - dimensional

$$
\begin{aligned}
y_i &= C_i x, \quad i \in \{1, 2, \ldots, m\} \\
\dot{x} &= Ax
\end{aligned}
$$

Agent i knows C_i and A and measures y_i

Joint observability: $\left(\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{bmatrix}, A \right)$ is an observable pair

$$C_i \neq 0 \quad i \in \{1, 2, \ldots, m\}$$

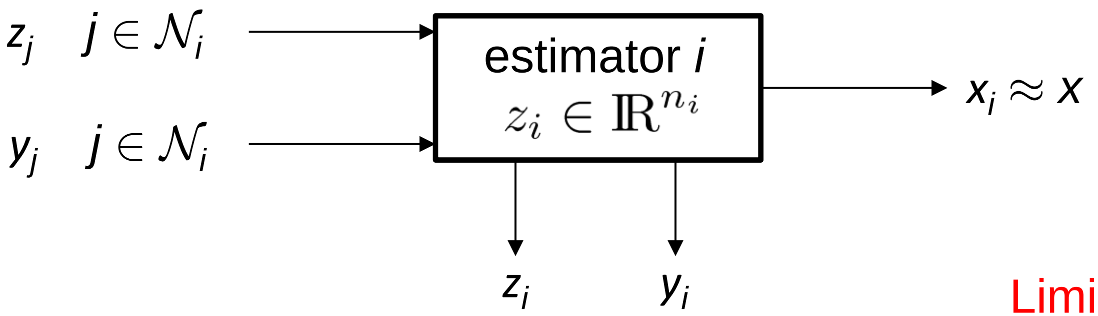
Objective: Devise a family of m estimators, one for each agent, whose outputs x_i, $i \in \{1,2,...,m\}$ are all asymptotically correct estimates of x

Suppose that each agent i generates estimate x_i as the output of an n_i - dimensional linear time invariant system with state z_i

Process

n - dimensional

$$y_i = C_i x, \quad i \in \{1, 2, \ldots, m\}$$
$$\dot{x} = Ax$$



Parks & Martins
TAC 2017

$$\dot{z}_i = \sum_{j \in \mathcal{N}_i} (H_{ij} z_j + K_{ij} y_j)$$

$$x_i = \sum_{j \in \mathcal{N}_i} (M_{ij} z_j + N_{ij} y_j)$$

Wang, Fullmer, Morse
ACC 2017

Limitations:

Cannot be used with most time-varying graphs

Not resilient

Objective: Devise a family of m estimators, one for each agent, whose outputs x_i, $i \in \{1,2,...,m\}$ are all asymptotically correct estimates of x

Suppose that each agent i generates estimate x_i as the output of an n_i - dimensional linear time invariant system with state z_i

Suppose that each agent i broadcasts its state z_i and measurement y_i

Hybrid Estimator

= local observers and linear equation solvers

n - dimensional

$$\begin{aligned} y_i &= C_i x, \quad i \in \{1, 2, \ldots, m\} \\ \dot{x} &= Ax \end{aligned}$$

Local Observer for Agent *i*

First construct a "classical observer" for each agent *i* which is capable of correctly estimating that "part of the process state," i.e., $L_i x$, which is observable to agent *i*:

Here L_i is any $n_i \times n$ matrix for which

$$\ker L_i = \ker \begin{bmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{n-1} \end{bmatrix} \qquad \text{rank } L_i = n_i \triangleq \text{rank} \begin{bmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{n-1} \end{bmatrix}$$

Let $(\bar{C}_i, \bar{A}_i)$ be the unique pair of matrices which satisfy the linear equations

$$C_i = \bar{C}_i L_i \qquad L_i A = \bar{A}_i L_i$$

and note that $(\bar{C}_i, \bar{A}_i)$ is an observable pair.

Choose K_i to make the convergence rate of $\bar{A}_i + K_i \bar{C}_i$ as large as desired.

Local observer *i*: $\boxed{\dot{w}_i = (\bar{A} + K_i \bar{C}_i) w_i + y_i}$

Estimation error: $\epsilon_i = w_i - L_i x$ satisfies $\epsilon_i = e^{(\bar{A}_i + K_i \bar{C}_i)t} \epsilon_i(0)$

$$w_i(t) = L_i x(t) + \epsilon_i(t) \qquad i \in \{1, 2, \ldots, m\}$$

Estimation error: $\epsilon_i = w_i - L_i x$ satisfies $\epsilon_i = e^{(\bar{A}_i + K_i \bar{C}_i)t} \epsilon_i(0)$

$$w_i(t) = L_i x(t) + \epsilon_i(t) \qquad i \in \{1, 2, \ldots, m\}$$

Pick a set of equally spaced "event times" t_0, t_1, t_2, ... where $t_0 = 0$ and $t_j - t_{j-1} = T$, $j \geq 1$.

Generate agent i's estimate $x_i(t)$ of $x(t)$ as follows:

1. For each fixed $j > 0$, iterate the previously discussed linear equation solver q times within the interval $[t_{j-1}, t_j)$ to obtain an estimate z_{ij} of the parameter p_j assuming p_j satisfies the equations

$$w_i(t_{j-1}) = L_i p_j, \quad i \in \{1, 2, \ldots, m\}$$

2. Prompted by the fact that

$$w_i(t_{j-1}) = L_i x(t_{j-1}) + \epsilon_i(t_{j-1}) \qquad i \in \{1, 2, \ldots, m\}$$

take z_{ij} to be an after the fact estimate of $x(t_{j-1})$ and define

$$\boxed{x_i(t_j) = e^{AT} z_{ij}}$$

3. Between event times generate $x_i(t)$ using

$$\boxed{\dot{x}_i = A x_i}$$

Can prove that with q properly chosen and with appropriate network connectivity {eg, strong connectivity $\forall$ t}, all $x_i(t)$ converge to $x(t)$ exponentially fast at any pre-specified rate. CDC 2017

Additional Properties

Robustness: The estimation is robust to small differences in agents' event times provided A is a stability matrix.

Asynchronism: Linear equation solver computations can be performed asynchronously.

Resilience: With enough redundancy the overall estimator is "resilient."

Resilience

By a *resilient algorithm* for a distributed process is meant an algorithm which, by exploiting built-in network and data redundancies, is able to continue to function correctly in the face of abrupt changes in the number of nodes and edges in the inter-agent communication graph upon which the algorithm depends.

Such changes might arise as a result of a network communication failure, a component failure, a sensor temporarily being put to sleep to conserve energy, or even possibly a malicious attack

Distributed estimators which are linear time-invariant systems are not resilient.

It is easy to see that the hybrid estimator just described, is.

Simulations

$$\dot{x} = Ax + bv \qquad v = \text{noise}$$

$$y_i = C_i x; \quad i \in \{1; 2; 3; 4\}$$

System: 4 agent, stable , 4-dimensional with eigenvalues at -0.1, -0.1 , -0.05 $\pm$ *j0.614*

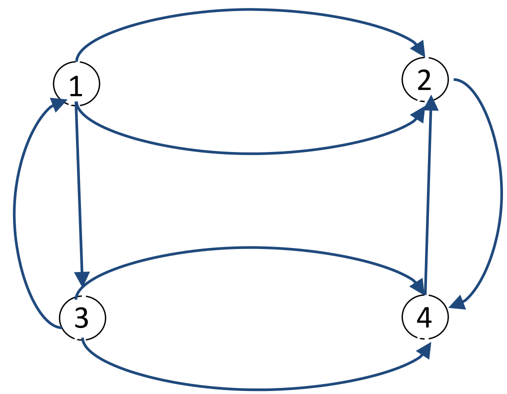
Simulations

$$\dot{x} = Ax + b\nu$$
$$y_i = C_i x;\ i \in \{1, 2, 3, 4\}$$

System: 4 agent, stable, 4-dimensional with eigenvalues at -0.1, -0.1 , -0.05 $\pm$ *j0.614*

redundantly jointly observable – remains jointly observable if one agent dies.

Neighbor graph:



Redundant strongly connected

Remains strongly connected if any one vertex is removed.

Not a complete graph

simulation with ν a sinusoid = 7cos(10 t)
simulation with ν white noise = {0 mean, variance 1}

In both simulations , agent 4 leaves the network at t = 5 and returns at t = 7

Simulations

ν = sine wave

$$\dot{x} = Ax + b\nu$$

$$y_i = C_i x; \quad i \in \{1; 2; 3; 4\}$$

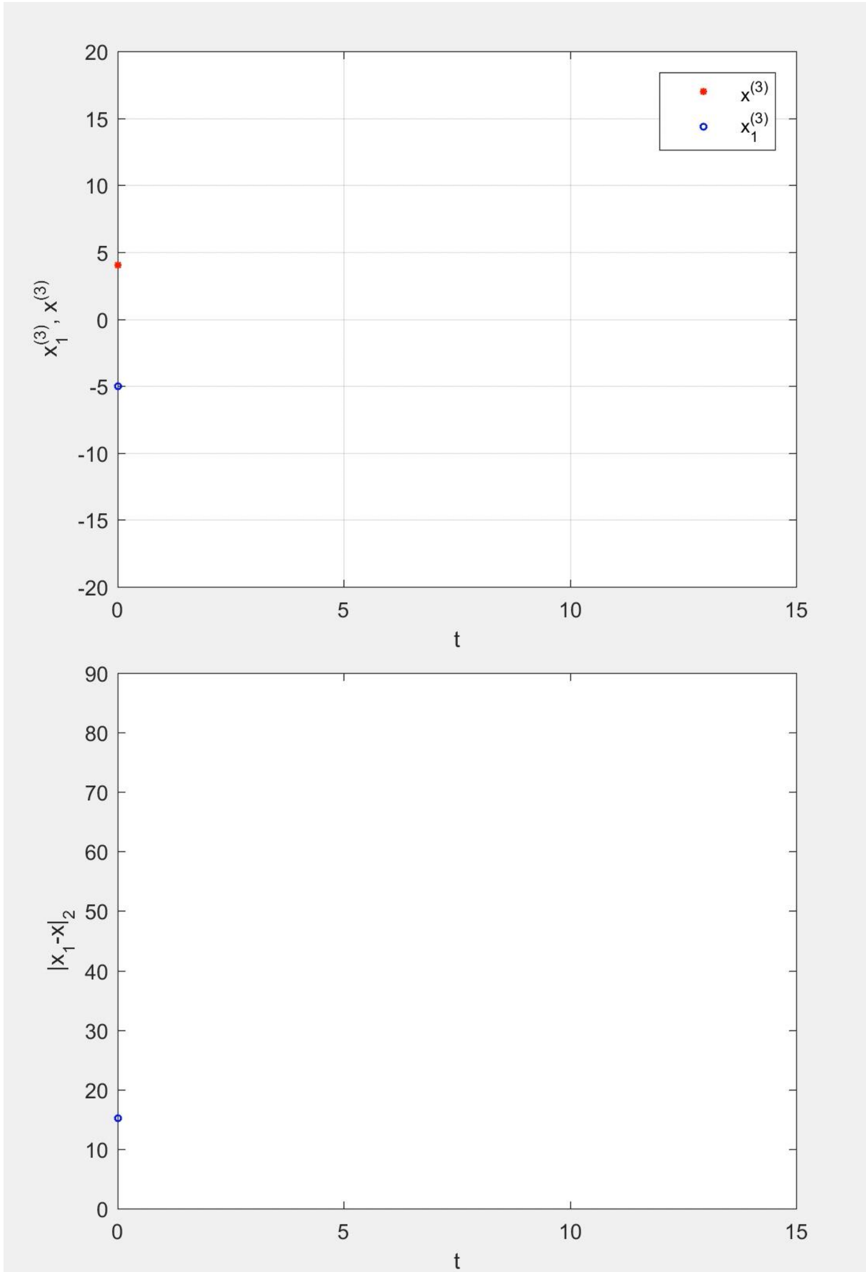
3rd components of x and x_1

$x_1^{(3)}$

$x^{(3)}$

state of estimator 1

$$||x_1 - x||_2$$



Agent 4 leaves the network at t = 5 and returns at t = 7

Simulations

ν = white noise

$$\dot{x} = Ax + b\nu$$

$$y_i = C_i x; \quad i \in \{1; 2; 3; 4\}$$

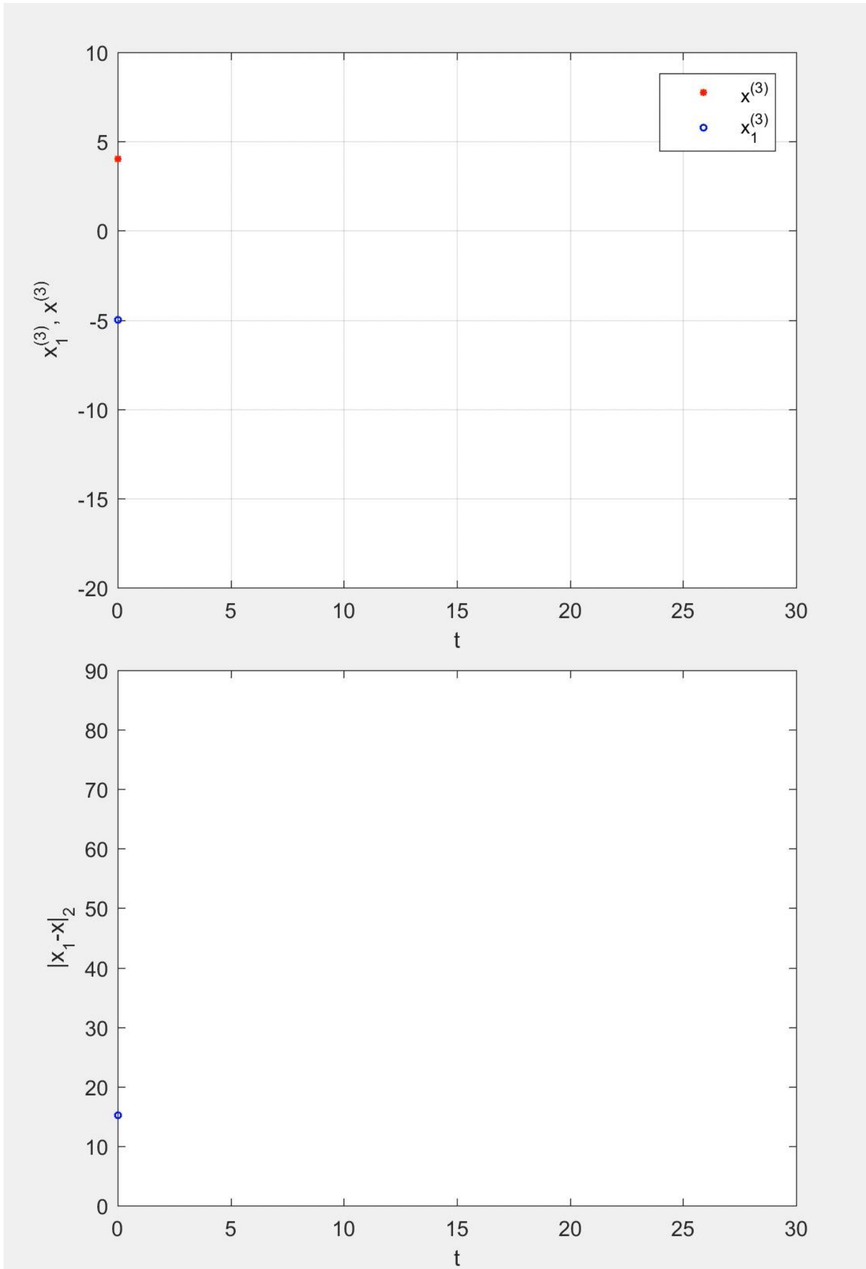
3rd components of x and x_1

$x_1^{(3)}$

$x^{(3)}$

state of estimator 1

$$||x_1 - x||_2$$



Agent 4 leaves the network at t = 5 and returns at t = 7

Why is Stability Needed for Robustness?

Process: $y = Cx$ $\dot{x} = Ax$ A = unstable

Estimator: $\dot{\widehat{x}} = \widehat{A}\widehat{x} + K(C\widehat{x} - y)$ $\hat{A} - A =$ small $\hat{A} + KC =$ stable

Estimation Error: $e = \widehat{x} - x$

$$\dot{e} = (\widehat{A} + KC)e + (\widehat{A} - A)x$$

Unbounded signal if $\widehat{A} \neq A$

Therefore for state estimation problems for processes without controlled inputs, internal stability must be assumed if the estimation problem is to make sense!

The same conclusion applies to distributed state estimation problems.

Concluding Remarks

Concerning para-contractions, it would be interesting to

1. derive necessary conditions on neighbor graph sequences for convergence
2. derive worst case bounds on convergence rate
3. discover other types of maps which are paracontractions

Concerning hybrid estimators, it would be interesting to

1. study what happends if different agents have different clocks
2. take into account transmission delays across the network

Concerning linear time invariant distributed observers it would be interesting to

1. prove that they cannot be resilient