Stability of Impulsive Hybrid Systems

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1. Impulsive Hybrid System (IHS)

1.1. Impulsive Phenomena
1. Impulsive Hybrid System (IHS)

1.1. Impulsive Phenomena

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   - Discrete-time IHS;
   - Stochastic IHS;
   - IHS with Time-delays; etc.
1. Impulsive Hybrid System (IHS)

1.1. Impulsive Phenomena

1.2. IHS Models:
   - Continuous-time IHS;
   - Discrete-time IHS;
   - Stochastic IHS;
   - IHS with Time-delays; etc.

1.3 IHS—Differential-Difference Inclusions
1.1. Impulsive Phenomena—911

911 Event in UAS

Fig.1.1. 911 Event.
1.1. Impulsive Phenomena—Bank Interest

Adjustment of Bank Interest Rate

Fig. 1.2. Adjustment of China Bank Interest Rates 1990–2008.
1.1. Impulsive Phenomena—MEMS Oscillators

MEMS (Micro-Electro-Mechanical System) Oscillator:

\[ m\ddot{y} + b\dot{y} + k_1 y + k_3 y^3 = v, \]  

where \( y \) is the displacement of the oscillator; \( m, b, k_1, k_3 \) are coefficients, typically \( m = 277, b = 0.678, k_1 = 7.61, k_3 = 0.441 \), and \( v \) is a driving force.

MEMS Oscillator with collisions.

\[ m\ddot{y} + b\dot{y} + k_1 y + k_3 y^3 = v, \quad t \neq t_k, \]  
\[ y(t_k) = y(t_k^-), \quad \dot{y}(t_k) = \dot{y}^-(t_k) + \Delta \dot{y}^-(t_k), \quad t = t_k, \]  

where \( \Delta \dot{y}^-(t_k) \) represents the change of velocity due to a collision at time \( t_k \).

Fig.1.3. Particles Collision.
1.2. Models–Continuous-time IHS

Continuous-time IHS:

\[ \dot{x} = f(t, x), \ t \neq t_k, \quad (4) \]
\[ \Delta x(t) = I_k(x(t)), \ t = t_k \quad (5) \]

where \( \Delta x(t) = x(t^+) - x(t) \).

- Impulses
  \( \{ \Delta x(t_k) = I_k(x(t_k)), k \in \mathbb{N} \} \).

- Impulsive instance sequence
  \( \{t_k, k \in \mathbb{N}\} \) satisfies:
  \[ 0 \leq t_0 < t_1 < \cdots < t_k < \cdots < \infty, \]
  \[ \lim_{k \to \infty} t_k = \infty. \]

- IHS is **discontinuous, right-continuous**
  (left-continuous if \( \Delta x(t) = x(t) - x(t^-) \)).

Figure of IHS.

Ref: Lakshmikantham, Bainov and Simeonor (1989), etc.
1.2. IHS—Discrete-time IHS (DIHS)

DIHS model (Li and X.Liu (03)):

\[ x(n + 1) = f(x(n)), \quad n \neq N_k, \]  \hspace{1cm} (6)
\[ \Delta x(n) = I_k(x(n)), \quad n = N_k \]  \hspace{1cm} (7)

where \( \Delta x(n) = x(n^+) - x(n) \), which means

\[ x(N_k + 1) = f(x(N_k) + I_k(x(N_k))). \]  \hspace{1cm} (8)

Impulsive instance sequence \( \{ N_k \in \mathbb{N}, k \in \mathbb{N} \} \) satisfies:

\[ 0 \leq N_0 < N_1 < \cdots < N_k < \cdots < \infty, \quad \lim_{k \to \infty} N_k = \infty. \]

Impulses happened in the interior of discrete-time system

\[ x(n + 1) = f(x(n)). \]
1.2. IHS—Discrete-time IHS (DIHS)

Revised DIHS model (B.Liu and X.Liu (07)):

\[ x(n + 1) = f(n, x(n)), \quad n \neq N_k, \]

\[ \Delta x(n) = I_k(x(n)), \quad n = N_k \]

where \( \Delta x(n) = x(n + 1) - x(n) \), which means

\[ x(N_k + 1) = x(N_k) + I_k(x(N_k)). \]

DIHS is a kind of **hybrid switching systems**: impulse results in switching between system (9) and (11).

Impulses can be looked as **external** input signals of discrete-time system \( x(n + 1) = f(x(n)) \).
1.2. IHS Models—Discrete-time IHS (DIHS)

DIHS provides a new control scheme for discrete-time systems:

\[
x(n + 1) = f(n, x(n)) + \sum_{k=1}^{\infty} \delta(n - N_k) g(n, x(n)),
\]

where controller \( u(n) = \sum_{k=1}^{\infty} \delta(n - N_k) g(n, x(n)) \), \( \delta(0) = 1 \) and \( \delta(n) = 0 \) for all \( n \neq 0 \).

Fig.1.5. Impulsive control for Discrete-time System
1.2. IHS Models—SIHS

SIHS (Stochastic Impulsive Hybrid Systems):

\[
\begin{align*}
\text{d}x(t) &= f(t, x(t))\text{d}t + g(t, x(t))\text{d}B(t), \quad t \in (t_k, t_{k+1}], \\
\Delta x(t) &= I_k(t, x(t)), \quad t = t_k, \ k \in \mathbb{N},
\end{align*}
\]

where \( B(t) = (B_1(t), \cdots, B_m(t))^T \) is an \( m \)-dimensional Brownian motion defined on the complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \) with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions (i.e., right continuous and \( \mathcal{F}_0 \) containing all \( P \)-null sets).

Ref: Yang and Xu (06); B. Liu and Liao (07); B.Liu (08); Wu and Sun (08).

General case of (13): \( \{t_k, k \in \mathbb{N}\} \) is a stochastic process, and \( I_k(t_k, x(t_k)), k \in \mathbb{N} \) are stochastic functions.
1.3. IHS—Differential-Difference Inclusions

Differential-Difference Inclusions:

\[
\dot{x} \in F(x), \quad x \in C, \\
x^+ \in G(x), \quad x \in D,
\]

(14)

where \( C \subset \mathbb{R}^n \), \( D \subset \mathbb{R}^n \), and \( F, G \) are set-valued functions:

\( F(x) \subset \mathbb{R}^n \), for any \( x \in C \) and

\( G(x) \subset \mathbb{R}^n \) for any \( x \in D \).

IHS (14) is more general hybrid systems, including switching systems.

Ref: Teel, Cai, and Goebel, et al (06,07,08).

Fig.1.6. Figure of IHS (14).
1.3. IHS—Differential-Difference Inclusions

An example: Bouncing Ball. (Teel et al (08))

\[
\begin{align*}
\dot{x} &= (-\gamma, x_1)^T, \; x \in C, \\
 x^+ &= -\rho x, \; x \in D,
\end{align*}
\] (15)

where $\rho \in (0, 1)$, $\gamma > 0$, $C = \{x \in \mathbb{R}^2 : x_2 \geq 0\}$, $D = \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 = 0\}$.

Fig.1.7. Figure of IHS (15).
2. Stability of IHS

2.1. Lyapunov Direct Method
2. Stability of IHS

2.1. Lyapunov Direct Method

2.2. Stability for IHS
2. Stability of IHS

2.1. Lyapunov Direct Method

2.2. Stability for IHS

2.3. Stability for IHS with Time-delays
2. Stability of IHS

2.1. Lyapunov Direct Method

2.2. Stability for IHS

2.3. Stability for IHS with Time-delays
2.1. Lyapunov Direct Method

Aleksandr Mikhailovich Lyapunov (1857–1918)


Question to Lyapunov:
It is known that at a certain angular velocity ellipsoidal forms cease to be the forms of equilibrium of a rotating liquid. In this case, do they not shift into some new forms of equilibrium which differ little from ellipsoids for small increases in the angular velocity?

(by Chebyshev in 1882)
2.1. Lyapunov Direct Method

Consider the general dynamical systems:

\[ \dot{x} = f(t, x), \quad t \geq t_0, t \in \mathbb{R}_+, \]
\[ x_0 = x(t_0), \quad (16) \]

where \( x \in \mathbb{R}^n \), \( f \) satisfies all required conditions such that solution exists uniquely on \( \mathbb{R}_+ \) and \( f(t, 0) \equiv 0 \), i.e., \( x = 0 \) is the equilibrium.

Lyapunov Function \( V(t, x) \)

- \( V \) is a continuous positive definite function:

\[ V(t, x) > 0, \text{ for } x \neq 0; \quad V(t, 0) \equiv 0. \]

“Energy” of the system
2.1. Lyapunov Direct Method

Lyapunov Stability Theorems:

Theorem 2.1. If there exists a positive definite function $V(t, x)$ such that for $\alpha \in K$,

$$\alpha(\|x\|) \leq V(t, x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n;$$  \hspace{1cm} (17)

$$\left. \frac{dV}{dt} \right|_{(16)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f \leq 0, \quad t \geq t_0,$$ \hspace{1cm} (18)

then, the equilibrium $x = 0$ of system (16) is stable.

Theorem 2.2. Assume that there exists a Lyapunov Function $V(t, x)$ such that for some $K_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$,

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n;$$ \hspace{1cm} (19)

$$\left. \frac{dV}{dt} \right|_{(16)} \leq -\alpha_3(V(t, x)), \quad t \geq t_0.$$ \hspace{1cm} (20)

Then, the equilibrium $x = 0$ of system (16) is asymptotically stable.
2.2. Stability of IHS

Consider the continuous-time IHS:

\[
\dot{x} = f(t, x), \quad t \neq t_k, \\
\Delta x = I_k(t, x), \quad t = t_k, k \in \mathbb{N},
\]

(21)

where \(x \in \mathbb{R}^n\), \(f, I_k\) satisfy all required conditions such that the solution to (21) exists uniquely on \(\mathbb{R}_+\) and \(f(t, 0) \equiv 0, I_k(t, 0) \equiv 0\), i.e., \(x = 0\) is the equilibrium of (21).

Lyapunov-like Function Class \(v_0\):

**Definition 2.1.** Let \(V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+\). \(V\) is said to belong to class \(v_0\) if

(i) \(V \in C[(t_{k-1}, t_k] \times \mathbb{R}^n, \mathbb{R}_+]\), \(k \in \mathbb{N}\) and for \((t, x) \in (t_{k-1}, t_k] \times \mathbb{R}^n\), it exists:

\[
\lim_{(t, y) \rightarrow (t_{k-1}^+, x), t > t_{k-1}} V(t, y) = V(t_{k-1}^+, x);
\]

(ii) \(V\) is locally Lipschitz in \(x\).
2.2. Stability of IHS

Question

How to take the derivative of $V \in v_0$?
2.2. Stability of IHS

Recall Dini derivative:

**Definition 2.2.** Let a function $V : \mathcal{O} \to \mathbb{R}$ (\(\mathcal{O}\) is open). For any $x \in \mathcal{O}$, define the *Dini derivative in the direction* $\xi \in \mathbb{R}^n$ as:

$$DV(x; \xi) = \lim_{w \to \xi, h \to 0^+} \sup \frac{V(x + hw) - V(x)}{h}.$$

**Remark 2.1.** By Rademacher’s Theorem and Clarke et al, (98), if $V$ is *locally Lipschitz*, then the gradient $\nabla V$ of $V$ exists at almost all $x \in \mathcal{O}$, and

$$DV(x; \xi) = \langle \nabla V(x), \xi \rangle.$$

Dini derivative of IHS:

**Definition 2.3.** For $(t, x) \in (t_{k-1}, t_k] \times \mathbb{R}^n$, we define:

$$D^+V(t, x) = \lim_{h \to 0^+} \sup \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)].$$
2.2. Stability of IHS

Three kinds of typical stability properties: Type-I, Type-II, Type-III

Type-I: stable continuous flow + stable impulses:

Theorem 2.3. If there exists a Lyapunov-like function $V \in v_0$ such that

\[
\phi_1(\|x\|) \leq V(t, x) \leq \phi_2(\|x\|); \tag{22}
\]

\[
D^+ V(t, x)_{(21)} \leq -\alpha(V(t, x)), \ t \neq t_k, k \in N; \tag{23}
\]

\[
\Delta V(t, x) \leq -\beta(V(t, x)), \ t = t_k, k \in N. \tag{24}
\]

Then, IHS (21) is asymptotically stable.

Stability properties–Type-I: $D^+ V < 0, \Delta V < 0$. 

\[\]
2.2. Stability of IHS

Type-II: unstable continuous flow + stable impulses

Theorem 2.4. There exists a function $V \in v_0$ satisfying (22) and functions $p \in PC[\mathbb{R}_+, \mathbb{R}_+]$, $c \in K$, $\psi \in C[\mathbb{R}_+, \mathbb{R}_+]$, constants $\gamma_i \geq 0$, $i \in \mathbb{N}$, such that

$$D^+ V(t, x)\big|_{(21)} \leq p(t)c(V(t, x)) \quad t \neq t_i;$$

$$V(t_i^+, x + I_i(x)) \leq \psi_i(V(t_i, x)), \quad i \in \mathbb{N};$$

$$\int_{t_i}^{t_{i+1}} p(s)ds + \int_s^{\psi_i(s)} \frac{ds}{c(s)} \leq -\gamma_i, \quad i \in \mathbb{N},$$

$$\sum_{i=1}^{\infty} \gamma_i = \infty.$$

Then IHS (21) is asymptotically stable.

Impulses can be used to stabilize system.
2.2. Stability of IHS

Stability properties–II: \( p_i \Delta_i < q_i \).

Remark 2.2. Maximal Dwell Time

\[
\Delta_{max} \equiv \sup_{i \in \mathbb{N}} \{ \Delta_i \} \leq \inf_{i \in \mathbb{N}} \left\{ \frac{-\int_s^{\psi_i(s)} \frac{d s}{c(s)} - \gamma_i}{p} \right\}
\]

where \( \Delta_i \equiv t_{i+1} - t_i, i \in \mathbb{N} \), and \( p(t) \leq p \) for some \( p > 0 \).
2.2. Stability of IHS

Type-III: stable continuous flow + unstable impulses:

Theorem 2.5. There exists a function $V \in v_0$ satisfying (22) and functions $p \in PC[\mathbb{R}_+, \mathbb{R}_+]$, $\psi \in C[\mathbb{R}_+, \mathbb{R}_+]$, constants $\gamma_i \geq 0$, $i \in \mathbb{N}$, such that

$$D^+ V(t, x) \leq -p(t) c(V(t, x)) \quad t \neq t_i, i \in \mathbb{N};$$  \hfill (29)

$$V(t_i^+, x + I_i(x)) \leq \psi_i(V(t_i, x)), \quad i \in \mathbb{N};$$  \hfill (30)

$$- \int_{t_i}^{t_{i+1}} p(s) ds + \int_s^{\psi_i(s)} \frac{ds}{c(s)} \leq -\gamma_i, \quad i \in \mathbb{N},$$  \hfill (31)

$$\sum_{i=1}^{\infty} \gamma_i = \infty.$$  \hfill (32)

Then, IHS (21) is asymptotically stable.
2.2. Stability of IHS

Stability properties—III: \( q_i < p_i \Delta_i \).

Remark 2.3. Minimal Dwell Time:

\[
\Delta_{\text{min}} \triangleq \inf_{i \in \mathbb{N}} \{ \Delta_i \} \\
\geq \sup_{i \in \mathbb{N}} \left\{ \int_{s_i}^{s_{i+1}} \frac{ds}{c(s)} + \gamma_i \right\},
\]

where \( p(t) \geq p \), for some \( p > 0 \).

Ref:

[Lakshmikantham, Bainov and Simeonov, (89); X.Liu, et al (94); Shen, et al (98), Ballinger, et al (01); B.Liu, et al (03,04); Michel, et al (98, 99, 05); etc.]

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2.3. Stability of IHS with Time-Delays

Consider IHS with *time-delays*:

\[
\dot{x} = f(t, x_t), \quad t \neq t_k, \\
\Delta x = I(t, x_{t-}), \quad t = t_k, \quad k \in \mathbb{N},
\]

where \( x_t \) satisfies \( x_t(\theta) = x(t + \theta), \quad \forall \theta \in [-\tau, 0], \) and \( x_0 = x(t_0) = \phi \in C([-\tau, 0], \mathbb{R}^n] \).

Special case (*differential-difference equation*):

\[
\dot{x} = f(t, x(t - \tau)), \quad t \neq t_k, \\
\Delta x = I(t, x_{t-}), \quad t = t_k, \quad k \in \mathbb{N}.
\]

Methods for stability w.r.t. time-delay:

- Lyapunov-Krasovskii functional method
- Razumikhin Technique
2.3. Stability of IHS with Time-Delays

Lyapunov-Krasovskii functional method

Lyapunov-Krasovskii functional is often in form of

\[ V(x) = x^T P x + \int_{t-\tau}^{t} \int_{t+\theta}^{t} x^T(s) Q x(s) ds + \int_{t-\tau}^{t} \int_{t+\theta-\tau}^{t} x^T(s) Z x(s) ds, \]

(35)

where matrices \( P > 0, Q \geq 0, Z \geq 0 \).

Suitable for systems with form of Lur’e Systems:

\[ \dot{x}(t) = Ax(t) + B x(t - \tau_1) + g(x(t - \tau_2)), \]

(36)

where \( g \) satisfies \( \| g(x) \| \leq \sum_{i=1}^{n} c_i \| x_i \| \) for some constants \( c_i \).

Delay-dependent stability criteria

2.3. Stability of IHS with Time-Delays

**Razumikhin Technique**

Relax the decreasing requirement on Lyapunov functional $V$:

\[
\dot{V} < 0 \text{ whenever } V(\varphi(0)) \geq g(\varphi(s)), \forall s \in [-\tau, 0], g \in \mathcal{K}.
\]

![Diagram showing stability analysis](image)

**Fig.2.4. Liu and Hill (09).**

**Ref:** For the case of continuous-time systems: Hale and Lunel, (1993); discrete-time systems: Liu and Marquez (07); Liu and Hill (09).
2.3. Stability of IHS with Time-Delays

Theorem 2.6. Assume $\Delta_{max} = \sup_{i \in N} \{t_{i+1} - t_i\} < \infty$ and there exist functions $V \in \nu_0$, and $c, g \in \mathcal{K}$ such that

(i) $\phi_1(\|x\|) \leq V(t, x) \leq \phi_2(\|x\|)$;

(ii) $D^+ V(t, \psi(0)) \leq p(t)c(V(t, \psi(0)))$, for all $t \neq t_i, \psi \in PC([-\tau, 0], R^n)$, if $V(t, \psi(0)) \geq g(t + s, \psi(s)), s \in [-\tau, 0]$;

(iii) $V(t_i, \psi(0) + I(t_i, \psi)) \leq g(V(t_i^-, \psi(0))), i \in N$;

(iv) $\sup_{t \in R_+} \int_t^{t+\Delta_{max}} p(s)ds < \inf_{q>0} \int_q^q \frac{ds}{c(s)}$.

Then, the IHS (34) is asymptotically stable.
2.3. Stability of IHS with Time-Delays

Theorem 2.7. Assume $\Delta_{\text{min}} = \inf_{i \in \mathbb{N}} \{t_{i+1} - t_i\} > 0$ and there exist functions $V \in v_0; c, g, \hat{g} \in \mathcal{K}$ with $s \leq \hat{g}(s) < g(s)$ for all $s > 0$ such that

\begin{enumerate}[(i)]
  \item $\phi_1(\|x\|) \leq V(t, x) \leq \phi_2(\|x\|)$;
  \item $D^+ V(t, \psi(0)) \leq -p(t)c(V(t, \psi(0)))$, for all $t \neq t_i, \psi \in PC([-\tau, 0], \mathbb{R}^n)$, if $g(V(t, \psi(0))) \geq V(t + s, \psi(s)), s \in [-\tau, 0]$;
  \item $V(t_i, \psi(0) + I(t_i, \psi)) \leq \hat{g}(V(t_i^-, \psi(0)))$, $i \in \mathbb{N}$;
  \item $\inf_{t \in \mathbb{R}^+} \int_t^{t+\Delta_{\text{min}}} p(s)ds > \sup_{q > 0} \int_q^g \frac{ds}{c(s)}$.
\end{enumerate}

Then, the IHS (34) is asymptotically stable.

Ref: Ballinger and X.Liu (01); Shen, et al (98,06); B.Liu, et al (05,07); Michel et al (05,07); Sun et al (05).
3. \(\mathcal{K}\mathcal{L}\)-stability of IHS

3.1. \(\mathcal{K}\mathcal{L}\)-stability of dynamical systems
3. $KLL$-stability of IHS

3.1. $KL$-stability of dynamical systems

3.2. Notions on $KLL$-stability
3. $KLL$-stability of IHS

3.1. $KL$-stability of dynamical systems

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3. $\mathcal{KLL}$-stability of IHS

3.1. $\mathcal{KLL}$-stability of dynamical systems

3.2. Notions on $\mathcal{KLL}$-stability

3.3. $\mathcal{KLL}$-stability of IHS

3.4. Open questions
Consider differential inclusions:

\[ \dot{x}(t) \in F(x(t)), \quad t \geq 0, \tag{37} \]

where \( x \in \mathbb{R}^n \), and set-valued function \( F \) satisfies hypothesis \((H)\):

- (H1) \( F(x) \) is a nonempty compact convex subsets \( \mathbb{R}^n \) for every \( x \in \mathbb{R}^n \).
- (H2) \( F \) is upper semicontinuous, i.e., given \( x \in \mathbb{R}^n \), for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[ \|x - x'\| < \delta \implies F(x') \subset F(x) + \epsilon B, \tag{38} \]

where \( B \) denotes the open unit ball.

**Definition 3.1. \( KL \)-function:**

- \( \mathcal{K} \)-function: A function \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is of class-\( \mathcal{K} \) (\( \gamma \in \mathcal{K} \)) if it is continuous, strictly increasing, and \( \gamma(0) = 0 \). \( \mathcal{K}_\infty \)-function: It is of class-\( \mathcal{K}_\infty \) if it is of class-\( \mathcal{K} \) and is unbounded.

- \( \mathcal{K}L \)-function: A continuous function \( \beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is of class-\( \mathcal{K}L \) if \( \beta(\cdot, t) \) is of class-\( \mathcal{K} \) for \( t \geq 0 \) and \( \beta(s, \cdot) \downarrow 0 \) for any \( s > 0 \).
3.1. $\mathcal{KL}$-stability of dynamical systems

**Definition 3.2.** $\mathcal{KL}$-stability: The system (37) is said to be $\mathcal{KL}$-stable if for any solution $x(t)$ to (37) with $x(0) = x_0$, there exists $\beta \in \mathcal{KL}$ such that

$$
\|x(t)\| \leq \beta(\|x_0\|, t), \quad t \geq 0.
$$

(39)

**$\mathcal{KL}$-stability Theorem:** ([Lin, Sontag and Wang, (96); Clarke, Ledyaev and Stern, (98)](98)) Theorem 3.1. The following are equivalent:

(i) System (37) is $\mathcal{KL}$-stable.

(ii) System (37) is uniformly asymptotically stable.

(iii) There exists a $C^\infty$-smooth Lyapunov function $V$ satisfying:

$$
\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^n;
$$

(40)

$$
\max_{v \in F(x)} \langle \nabla V(x), v \rangle \leq -\alpha_3(V(x)).
$$

(41)

where $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$. 
3.2. Notions on $KL\mathcal{L}$-stability

Differential-Difference Inclusions:

\[ \dot{x} \in F(x), \quad x \in C, \]
\[ x^+ \in G(x), \quad x \in D, \quad (42) \]

Hybrid Basic Conditions:
The state space $\mathcal{O}$ is open. The system (42) satisfies:

(A1): $C \subset \mathcal{O}$ and $D \subset \mathcal{O}$ are relatively closed in $\mathcal{O}$;

(A2): the (set-valued) map $F: \mathcal{O} \to \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in \mathcal{O}$;

(A3): the (set-valued) map $G: \mathcal{O} \to \mathbb{R}^n$ is outer semicontinuous and locally bounded, and for each $x \in D$, $G(x)$ is nonempty subset of $\mathcal{O}$.

Question:
Stability of a compact set $\mathcal{A}$ satisfying $\mathcal{A} \subset \mathcal{O}$?
3.2. Notions on $\mathcal{KLL}$-stability

**Proper indicator:**

**Definition 3.3.** The continuous function $\omega: \mathcal{O} \to \mathbb{R}_+$ is a **proper indicator** for $\mathcal{A}$ if it satisfies

$$\omega(x) = 0 \iff x \in \mathcal{A}.$$ 

Typically: $\omega(x) = d(x, \mathcal{A}).$

**Hybrid time domain $(\mathcal{I}, \{j\})$ and hybrid time $(t, j)$:**

- Hybrid time domain $(\mathcal{I}, \{i\})$: if $\mathcal{I} = [s, t) \subseteq \mathbb{R}_+$ and $i \in \mathbb{N}$.
- Hybrid time (variable) $(t, j)$: if $(t, j) \in (\mathcal{I}, \{i\})$, i.e., $t \in \mathcal{I}, j = i$, and $t$: the first time variable, $j$: the second time variable.

Let $\{t_k, k \in \mathbb{N}\}$ be the impulsive instance sequence in the IHS, then $([t_i, t_{i+1}), \{j\}), i, j \in \mathbb{N}$, are the hybrid time domains.

**Solution $\phi(t, j, x)$ to IHS (42):** Trajectory $\phi(t, j, x)$ starts from $x \in C \cup D$ satisfying (42) and $\phi(0, 0, x) = x$. 

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3.2. Notions on \( \mathcal{KLL} \)-stability

\textbf{\( \mathcal{KLL} \)-function:}

Definition 3.4. Let \( \beta(r, s, t) \in C[\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+] \). If for a given \((s, t)\), \(\beta(\cdot, s, t) \in \mathcal{K}\), i.e., it is strictly increasing w.r.t. \(r\) and \(\beta(0, s, t) = 0\); and for a given \(t\), \(\beta(\cdot, \cdot, t) \in \mathcal{KL}\), i.e., it is strictly decreasing w.r.t. \(s\) and \(\lim_{s \to \infty} \beta(r, s, t) = 0\); for a given \(s\), \(\beta(\cdot, s, \cdot) \in \mathcal{KL}\).

\textbf{\( \mathcal{KLL} \)-stability:}

Definition 3.5. Let \( \omega : \mathcal{O}_1 \to \mathbb{R}_+ \) be continuous. The set \( \mathcal{A} \) is said to be \( \mathcal{KLL} \)-stable w.r.t. \( \omega \) on \( \mathcal{O} \) if it is forward complete on \( \mathcal{O} \), and there exists \( \gamma \in \mathcal{KLL} \) such that for each \( x \in \mathcal{O} \), all solutions \( \phi \) starting from \( x \) satisfy

\[
\omega(\phi(t, j, x)) \leq \gamma(\omega(x), t, j), \quad \forall (t, j) \in \text{dom}\phi.
\]
3.2. Notions on $\mathcal{KLLL}$-stability

**Smooth Lyapunov Function of IHS:**

**Definition 3.6.** Let $\mathcal{O}_1 \subset \mathcal{O}$ be open and $\omega : \mathcal{O}_1 \to \mathbb{R}_+$ be continuous. A function $V : \mathcal{O}_1 \to \mathbb{R}_+$ is said to be a *smooth Lyapunov function of IHS (42)* if it is smooth and there exist class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2$ such that

$$
\alpha_1(\omega(x)) \leq V(x) \leq \alpha_2(\omega(x)), \quad \forall x \in \mathcal{O}_1, \quad (43)
$$

$$
\max_{f \in F(x)} < \nabla V(x), f > \leq -V(x), \quad \forall x \in \mathcal{O}_1 \cap C, \quad (44)
$$

$$
\sup_{g \in G(x) \cap \mathcal{O}_1} V(g) \leq e^{-1} V(x) < V(x), \quad \forall x \in x \in \mathcal{O}_1 \cap D. \quad (45)
$$

**Remark 3.1.** Inequality (44) implies that $V$ is strictly decreasing on the continuous-time; and (45) implies that $V$ is also strictly decreasing on the discrete-time.
3.2. Notions on $KLL$-stability

robustly $KLL$-stable

- Admissible perturbation radius $\sigma$:

  **Definition 3.7.** A continuous function $\sigma : \mathcal{O} \to \mathbb{R}_+$ is said to be an *admissible perturbation radius* if $\{x\} + \sigma(x)\overline{B} \subset \mathcal{O}$ for all $x \in \mathcal{O}$.

- $\sigma$-perturbation of IHS (42):

\[
\begin{align*}
\dot{x} &\in F_\sigma(x), \quad \forall x \in C_\sigma, \\
x^+ &\in G_\sigma(x), \quad \forall x \in D_\sigma,
\end{align*}
\] (46)

where

\[
\begin{align*}
F_\sigma(x) &= \overline{\text{co}}F((x + \sigma(x)\overline{B}) \cap C) + \sigma(x)\overline{B}, \forall x \in \mathcal{O}; \\
G_\sigma(x) &= \{v \in \mathcal{O} : v \in g + \sigma(g)\overline{B}, g \in G((x + \sigma(x)\overline{B}) \cap D)\}, \forall x \in \mathcal{O}; \\
C_\sigma &= \{x \in \mathcal{O} : (x + \sigma(x)\overline{B}) \cap C \neq \emptyset\}; \\
D_\sigma &= \{x \in \mathcal{O} : (x + \sigma(x)\overline{B}) \cap D \neq \emptyset\}.
\end{align*}
\]
3.3. $\mathcal{KLL}$-stability of IHS

**Definition 3.8.** If there exists an admissible perturbation radius $\sigma$ such that IHS (46) is $\mathcal{KLL}$-stable, then, we say the IHS (42) is robustly $\mathcal{KLL}$-stable.

**Main Results:** (Teel, Cai, and Goebel, et al (06,07,08))

**Theorem 3.2.** Suppose $C \cup D = \mathcal{O}$. Let $\omega$ be any proper indicator of compact $\mathcal{A}$ w.r.t. $\mathcal{O}$ and the solution to (42) is forward complete. Then, the following are equivalent:
(i) the set $\mathcal{A}$ is asymptotically stable for (42);
(ii) there exists a smooth Lyapunov function $V$ of (42);
(iii) the system (42) is robustly $\mathcal{KLL}$-stable w.r.t. $\omega$ on $\mathcal{O}$.

**Remark 3.2.** $\mathcal{KLL}$-stable:
\[
\omega(\phi(t, j, x)) \leq \gamma(\omega(x), t, j) \leq \alpha_1^{-1}(\alpha_2(\omega(x))e^{-(t+j)}), \quad \forall (t, j) \in \text{dom}\phi.
\]

- Strong stability property
- smooth Lyapunov function $V$
3.4. Open questions

Consider

\[ \dot{x} \in F(x), \]

where \( F \) satisfies hypothesis \((H)\) and is locally Lipschitz.

Question 1: It is \( KL \)-stable if and only if there exists a locally Lipschitz (not necessarily smooth) Lyapunov function \( V \) for \( F \)?

Converse Theorem: (Clarke, Ledyaev and Stern, (98))

If \( F \) is \( KL \)-stable, then there exits a locally Lipschitz strong Lyapunov function \( V \) for \( F \).

Consider IHS:

\[ \dot{x} \in F(x), \quad x \in C, \]

\[ x^+ \in G(x), \quad x \in D, \quad (*) \]

where \( F, G \) satisfy locally Lipschitz and Hybrid Basic Conditions.
3.3. Open questions

**Question 2:** IHS (*) is $K\mathcal{CL}$-stable if and only if there exists a locally Lipschitz (not necessarily smooth) Lyapunov-like function $V \in v_0$ satisfying (43)-(45)?

**Ref:** Lyapunov Converse Theorems

- Lyapunov (1892)
- Kurzweil (1956)
- Krasovskii, (1963)
- Fillippov, (1962, 1988)
- Zubov, (1964)
- Yoshizawa, (1966)
- Sontag, (1983)
- Clarke, Ledyaev and Stern, (1998)
- Teel and Praly, (00);
- Kellett and Teel (05); Goebel and Teel, (06); Cai and Teel (07,08)
4. Examples–Impulsive stabilization of chaotic systems

Example 4.1. Impulsive stabilization of chaotic systems
Consider Lorenz system:

\[ \dot{x} = Ax + \varphi(x) \]  \hspace{1cm} (47)

where \( x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \), \( A = \begin{pmatrix} -b_1 & b_1 & 0 \\ b_2 & -1 & 0 \\ 0 & 0 & -b_3 \end{pmatrix} \), \( \varphi(x) = (0, -x_1 x_3, x_1 x_2)^T \).

It is well-known that under \( b_1 = 10, b_2 = 28, b_3 = \frac{8}{3} \), the Lorenz system (47) is chaotic.

Aim: Design impulsive control \( \{(u(t), t_k) : k \in \mathbb{N}\} \) to stabilize system (47).
4. Examples–Impulsive stabilization of chaotic systems

**Impulsive stabilization of chaotic systems**

**Impulsive control scheme:**

\[
    u(t, x) = \sum_{k=1}^{\infty} \delta(t - t_k) B_k x(t),
\]

(48)

such that

\[
    \dot{x} = Ax + \varphi(x) + u(t, x),
\]

(49)

is asymptotically stable, where \( B_k \in \mathbb{R}^{n \times n}, k \in \mathbb{N} \), are impulsive control gain matrices, and \( \{t_k, k \in \mathbb{Z}\} \) are the impulsive instances, and \( \delta(\cdot) \) is the Dirac delta function.

**Impulsive control** \( \{(B_k, t_k) : k \in \mathbb{N}\} \) to be designed.
4. Examples–Impulsive stabilization of chaotic systems

Design impulsive control \( \{(B_k, t_k) : k \in \mathbb{N}\} \)

Taking Lyapunov function \( V \) as:

\[
V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)
\]

Then, it is easy to get

\[
D^+ V \leq \cdots \leq \alpha V, \quad t \neq t_k, k \in \mathbb{N}.
\] (50)

where \( \alpha = 33.0512 > 0 \).

\( \{(B_k, t_k) : k \in \mathbb{N}\} \) can be chosen as (by Theorem 2.4):

\[
B_k = b_k I, \quad -0.5 < b_k < \frac{1}{2}\{e^{-\gamma_k \cdot \alpha \Delta t_k} - 1\},
\] (51)

where \( \Delta t_k = t_{k+1} - t_k \) and \( \gamma_k \geq 0 \) with \( \sum_{k=0}^{\infty} \gamma_k = \infty \).
4. Examples–Impulsive stabilization of chaotic systems

Simulations:

![Phase portrait of Lorenz system.](image1)

**Fig.4.1.** Phase portrait of Lorenz system.

Impulsive control data:

\[ B_k = -0.3I, \Delta t_k = 0.2, k \in \mathbb{N}. \]

Impulsive stabilization:

![Phase portrait for impulsive stabilization of Lorenz system.](image2)

**Fig.4.2.** Phase portrait for impulsive stabilization of Lorenz system.

Ref. B. Liu, K.L. Teo and X. Liu, (04)
4. Examples—Impulsive Adjustment of Satellite Orbit

Satellite Collision—USA & Russia in Feb.11,2009.

Fig.4.3. Satellite Collision—2009.
4. Examples—Impulsive Adjustment of Satellite Orbit

Moon Probing Plan of China—(2017))

Fig.4.4. Moon Probing Plan of China—(2017).
4. Examples—Impulsive Adjustment of Satellite Orbit

Example 4.2. Impulsive Adjustment of Satellite Orbit

(Masutani, et al (01))

![Diagram of satellite orbit adjustment](image)

Fig. 1 Modeling of Flyaround Maneuver
4. Examples—Impulsive Adjustment of Satellite Orbit

- Impulsive Adjustment of Satellite Orbit

![Diagram of Elliptic Flyaround Maneuver](image)

**Fig. 2 Elliptic Flyaround Maneuver**
4. Examples—Impulsive Adjustment of Satellite Orbit

**Impulsive Adjustment of Satellite Orbit**

![Diagram of Bi-elliptic Flyaround Maneuver](image)

**Fig. 3 Bi-elliptic Flyaround Maneuver**
5. Conclusions

Three kinds of stability properties of IHS:

- Several IHS Models
- impulses have no effects to the stability;
- impulses destroy the stability;
- impulses do contribution to the stability.
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- Three kinds of stability properties of IHS:
  - Several IHS Models
  - impulses have no effects to the stability;
  - impulses destroy the stability;
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- Stability of IHS with time-delays:
  - Razumikhin-type stability theorem.
5. Conclusions

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  - impulses have no effects to the stability;
  - impulses destroy the stability;
  - impulses do contribution to the stability.

Stability of IHS with time-delays:

- Razumikhin-type stability theorem.

\( \mathcal{KLL} \)-stability of IHS:

- \( \mathcal{KLL} \)-stability conditions for differential inclusions
- Sufficient and necessary conditions for \( \mathcal{KLL} \)-stability
- Conservative and Open questions
5. Conclusions

Three kinds of stability properties of IHS:

- Several IHS Models
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Stability of IHS with time-delays:

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\( KLL \)-stability of IHS:

- \( K\ell \)-stability conditions for differential inclusions
- Sufficient and necessary conditions for \( KLL \)-stability
- Conservative and Open questions
6. Future Works on IHS

- More general stability for IHS
- Hybrid Time
- Discrete Dynamics in IHS
- Stochastic IHS

Stability and control of IHS with Time-delay
- Razumikhin-type Theorems for discrete-time IHS
- control with time-delay $u(t - \tau)$

ISS (Input-to-State Stability) of IHS
- IHS without time-delays
  - Ref: Liu and Marquez (07), Hespanha et al (08), Cai and Teel (09)
- IHS with time-delays
  - Ref: Liu and Hill (09)
Future Works on IHS

**KL-stability properties of IHS via hybrid time**

**KL**-stability:

\[ \|x(t, i, x_0)\| \leq \gamma(\|x_0\|, t, i), \quad (t, i) \in \mathcal{I}_i, i \in \mathbb{N}, \quad t \geq 0, \quad (52) \]

where for some \( \gamma \) satisfying \( \gamma \in \mathcal{KL} \).

**KL-stability w.r.t. the first time variable:**

\[ \|x(t, i, x_0)\| \leq \beta(\|x_0\|, t), \quad (t, i) \in \mathcal{I}_i, i \in \mathbb{N}, \quad t \geq 0. \quad (53) \]

where for some \( \beta \) satisfying \( \beta \in \mathcal{KL} \).

**KL-stability w.r.t. the second time variable:**

\[ \|x(t, i, x_0)\| \leq \beta(\|x_0\|, i), \quad (t, i) \in \mathcal{I}_i, i \in \mathbb{N}, \quad t \geq 0, \quad (54) \]

where for some \( \beta \) satisfying \( \beta \in \mathcal{KL} \).
6. Future Works on IHS

One simple example: Consider IHS as follows:

\[ \dot{x} = 0, \quad t \neq t_k; \]
\[ \Delta x = (q - 1)x(t), \quad t = t_k, \quad k \in \mathbb{N}, \tag{55} \]

where \( x \in \mathbb{R}, \ 0 < q < 1. \)

Then, the solution to IHS (55) satisfies

\[ \|x(t, i, x_0)\| \leq \|x_0\|q^i, \quad (t, i) \in I_i, \ i \in \mathbb{N}, \ t \geq 0. \tag{56} \]

Hence, IHS (55) has only KL-stability w.r.t. the second time variable, where \( \beta(s, t) = sq^t \) for any \( s, t \in \mathbb{R}_+. \)
Thank you!