Cascade Cavity Realization for a Class of Complex Transfer Functions arising in Coherent Quantum Feedback Control

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Introduction

- In recent years, there has been considerable interest in the feedback control of quantum systems. In particular, some recent papers have considered the case in which the feedback controller itself is also a quantum system. Such feedback control is often referred to as coherent quantum control.

- Due to the limitations imposed by quantum mechanics on the use of measurement, the use of coherent quantum feedback control may lead to improved control system performance. In addition, in many applications, coherent quantum feedback controllers may be preferable to classical feedback controllers due to considerations of speed and ease of implementation.

- A number of existing results on the synthesis of coherent quantum feedback control systems assume that the plant is a linear quantum system. Such linear quantum systems commonly arise in the area of quantum optics.
In order to apply these synthesis results on coherent quantum feedback controller synthesis, it is necessary to realize a synthesized feedback controller transfer function using physical optical components such as optical cavities, beam-splitters, optical amplifiers, and phase shifters.

In a recent paper by Nurdin James and Doherty, this issue was addressed for a general class of coherent linear quantum controllers. The paper by Nurdin James and Doherty presents an algorithm for realizing a given linear quantum controller using a series connection of optical cavities and optical amplifiers. The algorithm also requires direct couplings between the different optical cavities.
The main result of this paper is an alternative approach to the problem of physical realization for coherent linear quantum feedback controller transfer functions. Our approach only applies to the restricted class of complex linear quantum optical systems considered in a recent paper by Maalouf and Petersen. This class of linear quantum systems includes quantum systems made up purely of passive optical components such as cavities, beam-splitters and phase shifters.

Our approach to the realization of complex linear quantum systems involves the series interconnection of only passive optical cavities and involves only optical connections between the cavities.
Problem Formulation

Complex Quantum Systems
Consider a class of complex quantum optical systems described in terms of the complex annihilation operator by the quantum stochastic differential equations (QSDEs)

\[ da(t) = Fa(t)dt + Gdu(t); \]
\[ dy(t) = Ha(t)dt + Jdu(t) \]

where \( F \in \mathbb{C}^{n \times n}, \ G \in \mathbb{C}^{n \times m}, \ H \in \mathbb{C}^{m \times n} \) and \( J \in \mathbb{C}^{m \times m} \).

- Here \( a(t) = [a_1(t) \cdots a_n(t)]^T \) is a vector of (linear combinations of) annihilation operators.

- The vector \( u \) represents the input signals and is assumed to admit the decomposition:

\[ du(t) = \beta_u(t)dt + d\tilde{u}(t). \]
The noise $\tilde{u}(t)$ is a vector of quantum noises. The noise processes can be represented as operators on an appropriate Fock space.

The process $\beta_u(t)$ represents variables of other systems which may be passed to the system via an interaction.

**Definition.** The complex linear quantum system above is said to be a quantum system realization of a complex transfer function matrix $K(s)$ if

$$K(s) = H(sI - F)^{-1}G + J.$$
The aim of this paper is to begin with a complex transfer function matrix $K(s)$ and find a corresponding quantum system realization such that the quantum system can be physically realized via a cascade interconnection of optical cavities and phase shifters.
Definition. *The above complex linear quantum system is said to be physically realizable if there exists a commutation matrix* \( \Theta = \Theta^\dagger > 0 \), a coupling matrix \( \Lambda \), and a Hamiltonian matrix \( M = M^\dagger \) such that

\[
F = -\Theta \left( iM + \frac{1}{2} \Lambda^\dagger \Lambda \right); \quad G = -\Theta \Lambda^\dagger; \\
H = \Lambda; \quad J = I.
\]

Here, the notation \( ^\dagger \) represents complex conjugate transpose. In this definition, if the system is physically realizable, then the matrices \( M \) and \( \Lambda \) define a complex open harmonic oscillator with coupling operator \( L = \Lambda a \) and a Hamiltonian operator \( \mathcal{H} = a^\dagger Ma \).
**Definition.** The above complex linear quantum system is said to be lossless bounded real if

i) $F$ is a Hurwitz matrix; i.e., all of its eigenvalues have strictly negative real parts.

ii) The transfer function matrix $K(s) = H(sI - F)^{-1}G + J$ satisfies $K(i\omega)^\dagger K(i\omega) = I$ for all $\omega \in \mathbb{R}$.

**Lemma.** (Complex Lossless Bounded Real Lemma.) Suppose the above system is minimal. Then the system is lossless bounded real if and only if there exists a complex Hermitian matrix $X > 0$ such that

$$XF + F^\dagger X + H^\dagger H = 0;$$

$$H^\dagger J = -XG;$$

$$J^\dagger J = I.$$
The following lemma, contained in a recent paper by Maalouf and Petersen, shows the connection between the physical realizability property and the lossless bounded real property for the class of complex linear quantum systems under consideration.

**Lemma.** A minimal linear complex quantum system of the above form is physically realizable if and only if $J = \bar{I}$ and the system is lossless bounded real.
Cascade Cavity Realization

- An optical ring cavity consists of a number of partially reflecting mirrors arranged to produce a traveling light wave when coupled to a coherent light source.
Such a cavity with $m$ mirrors, can be described by a complex linear quantum system of the above form as follows:

$$da = \left(-\frac{\gamma}{2} + i\Delta\right) a dt - \sum_{i=1}^{m} \sqrt{\kappa_i} du_i;$$

$$dy_i = \sqrt{\kappa_i} a dt + du_i, \quad i = 1, 2, \ldots, m,$$

where $\gamma = \sum_{i=1}^{m} \kappa_i$ and $a$ is a single annihilation operator associated with the cavity mode.

The quantities $\kappa_i \geq 0, \quad i = 1, 2, \ldots, m$ are the coupling coefficients which correspond to the mirrors which make up the cavity. Also, the quantity $\Delta \in \mathbb{R}$ corresponds to the detuning between the resonant frequency of the cavity and the frequency of the coherent light source.
For each mirror in the optical cavity there corresponds an input field and an output field. If a particular mirror is not coupled to a coherent light source, it is modelled as being coupled to a vacuum noise source. Thus, an \( m \) mirror optical cavity acts as a first order, \( m \) input, \( m \) output linear complex quantum system.
We can generalize this system somewhat by introducing phase-shifters on the input and output channels. In practice, such phase shifts would correspond to adjusting the distances between the optical cavity and other elements in the quantum optical network to which the cavity is coupled.
The linear complex quantum system which describes this generalized cavity is as follows:

\[ da = \left( -\frac{\gamma}{2} + i\Delta \right) a dt - \sum_{i=1}^{m} \sqrt{\kappa_i} e^{-i\theta_i} du_i; \]

\[ dy_i = \sqrt{\kappa_i} e^{i\theta_i} a dt + du_i, \quad i = 1, 2, \ldots, m. \]
Thus, a generalized cavity can be described by quantum stochastic differential equations of the form:

\[
da = padt - h^\dagger du; \\
dy = hadt + du
\]

where

\[
p + p^* = -\gamma = -\sum_{i=1}^{m} \kappa_i = -h^\dagger h.
\]

Here \( p = -\gamma/2 + i\Delta, \)

\[
h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix} \quad = \quad \begin{bmatrix} \sqrt{\kappa_1} e^{i\theta_1} \\ \sqrt{\kappa_2} e^{i\theta_2} \\ \vdots \\ \sqrt{\kappa_m} e^{i\theta_m} \end{bmatrix}, \\
du = \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_m \end{bmatrix}, \\
dy = \begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_m \end{bmatrix}.
\]
It now follows that any first order complex linear quantum system of this form, with non-zero \( h \in \mathbb{C}^m \) and satisfying \( p + p^* = -h^\dagger h \), can be physically realized as a generalized \( m \) mirror cavity.

In this case, the mirror coupling coefficients and phase shifts are determined using a polar coordinates description of the elements of \( h \).

Also, the detuning parameter \( \Delta \) is determined from the imaginary part of the system pole \( p \).

Note that any complex linear quantum system of this form will be lossless bounded real (and hence physically realizable). Indeed, it is straightforward to verify that in this case, \( J = 1 \) and \( X = 1 \) satisfies the conditions of the Lossless Bounded Real lemma.
We now consider a cascade interconnection of $n$, generalized $m$ mirror cavities:

\[ \begin{align*}
\begin{array}{ccc}
\text{generalized} & \text{generalized} & \text{generalized} \\
\text{cavity 1} & \text{cavity 2} & \text{cavity n}
\end{array}
\end{align*} \]

$u = u_1 \Rightarrow u_2 = y_1 \Rightarrow \cdots \Rightarrow u_n = y_{n-1} \Rightarrow y = y_n$

In this cascade system, the $i$th cavity is described by the following complex linear quantum stochastic differential equations:

\[ \begin{align*}
da_i &= p_ia_i dt - H_i^\dagger du; \\
dy &= H_i a_i dt + du
\end{align*} \]

where

\[ p_i + p_i^* = -H_i^\dagger H_i. \]
The complete cascade system is described by the following complex linear quantum stochastic differential equations:

\[
\begin{align*}
\text{da}_1 &= p_1a_1dt - H_1^\dagger du; \\
\text{da}_i &= -\sum_{j=1}^{i-1} H_i^\dagger H_j a_j dt + p_i a_i dt - H_i^\dagger du; \\
\text{dy} &= H_i a_i dt + du.
\end{align*}
\]
The complete cascade system is a complex linear quantum system where

\[
F = \begin{bmatrix}
p_1 & 0 & \ldots & 0 \\
-H_2^\dagger H_1 & p_2 \\
\vdots & \ddots & \ddots & \vdots \\
-H_n^\dagger H_1 & \ldots & -H_n^\dagger H_{n-1} & p_n
\end{bmatrix},
\]

\[
G = -\begin{bmatrix}
H_1^\dagger \\
H_2^\dagger \\
\vdots \\
H_n^\dagger
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
H_1 & H_2 & \ldots & H_n
\end{bmatrix},
\]

\[
J = I.
\]
The aim of this paper is to begin with an arbitrary physically realizable quantum system and corresponding transfer function. Then to find if possible, a state space transformation such that the resulting transformed system is of the form a cascade of cavities and such that the condition $p_i + p_i^* = -H_i^\dagger H_i$ is satisfied for all $i$.

In this case, the system can be physically realized via a cascade of generalized cavities.

This quantum realization problem would typically arise when the transfer function and system represent a desired coherent quantum controller derived using a controller synthesis method such as quantum $H^\infty$ or quantum LQG control.
Lemma. Consider a cascade quantum system of the form given above satisfying condition $p_i + p_i^* = -H_i^\dagger H_i$ for all $i$. Then this system will satisfy the conditions of the Lossless Bounded Real Lemma with $X = I$.

Proof. To establish this result, we first consider the matrix $F + F^\dagger + H^\dagger H$. The $(i, i)$th element of this matrix is given by $p_i + p_i^* + H_i^\dagger H_i$ which is zero by assumption. Also, the $(i, j)$th element of this matrix with $j < i$ is given by $-H_i^\dagger H_j + H_i^\dagger H_j = 0$. Similarly, the $(i, j)$th element of this matrix with $j > i$ will be zero since the matrix is Hermitian. Thus, we can conclude that $F + F^\dagger + H^\dagger H = 0$. Also, note that the equations defining the matrices in the cascade quantum system imply $H^\dagger + G = 0$. Thus, the lossless bounded real conditions are satisfied with $X = I$. □
The Main Result

- The main result of this paper is an algorithm for realizing a physically realizable quantum system and transfer function via a cascade of generalized cavities.

- We restrict attention to quantum systems in which the eigenvalues of the matrix $F$ are all distinct.

- In this case, it follows via a (complex version of a) standard result from linear systems theory, that the linear complex quantum system can be transformed into Modal Canonical Form.
The complex linear quantum system in modal canonical form is assumed to be as follows:

\[
\begin{align*}
    d\tilde{a}(t) &= \tilde{F}\tilde{a}(t)dt + \tilde{G}du(t); \\
    dy(t) &= \tilde{H}\tilde{a}(t)dt + du(t)
\end{align*}
\]

where

\[
\begin{align*}
    \tilde{F} &= \begin{bmatrix} p_1 & 0 & \ldots & 0 \\
                 0 & p_2 & & \vdots \\
                 \vdots & & \ddots & 0 \\
                 0 & \ldots & 0 & p_n \end{bmatrix}; \\
    \tilde{G} &= \begin{bmatrix} \tilde{G}_1 \\
                                \tilde{G}_2 \\
                                \vdots \\
                                \tilde{G}_n \end{bmatrix}; \\
    \tilde{H} &= \begin{bmatrix} \tilde{H}_1 & \tilde{H}_2 & \cdots & \tilde{H}_n \end{bmatrix}.
\end{align*}
\]
Lemma. If the complex linear quantum system in modal canonical form is minimal, then each of the vectors $\tilde{H}_i$ must be non-zero.

Proof. If $\tilde{H}_i = 0$ for some $i$, then $\tilde{H}e_i = 0$ where $e_i$ is the standard unit vector with a 1 in the $i$th position and all other elements are zero. Furthermore, $\tilde{F}e_i = p_ie_i$. However, this contradicts the observability of the system and thus, we must conclude that $\tilde{H}_i \neq 0$ for all $i$. $\square$
We are now in a position to present the proposed algorithm:

**Step 1:** Begin with a minimal modal canonical form realization of the lossless bounded real transfer function $K(s)$.

**Step 2:** Let

\[
\bar{H}_n = \bar{H}_n, \alpha_n = -\frac{H_n^\dagger H_n}{p_n + p_n^*} ,
\]

\[
H_n = \frac{\bar{H}_n}{\sqrt{\alpha_n}}, \quad t(n, n) = \frac{1}{\sqrt{\alpha_n}}.
\]
Step 3: Calculate the quantities $H_n, H_{n-1}, \ldots, H_1$, 
$\alpha_n, \alpha_{n-1}, \ldots, \alpha_1, t(i, j)$, for $j = n, n - 1, \ldots, 1$ and $j \geq i$. 
These values are calculated using the following recursive formulas 
starting with the values determined in Step 2 for $i = n$:

$$
\bar{H}_i = \left[ I + \sum_{j=i+1}^{n} \frac{\bar{H}_j}{p_j - p_i} \sum_{k=i+1}^{j} t(j, k) H_k^\dagger \right]^{-1} \tilde{H}_i;
$$

$$
\alpha_i = -\frac{\bar{H}_i^\dagger \bar{H}_i}{p_i + p_i^*}, \quad H_i = \frac{\bar{H}_i}{\sqrt{\alpha_i}},
$$

$$
t(k, i) = \frac{1}{p_i - p_k} \sum_{j=i+1}^{k} t(k, j) H_j^\dagger H_i \text{ for } k = i + 1, \ldots, n,
$$

$$
t(i, i) = \frac{1}{\sqrt{\alpha_i}}.
$$
Step 4: Set $t(k, i) = 0$ for $k > i$ and define an $n \times n$ transformation matrix $T$ whose $(i, j)$th element is $t(i, j)$.

We now present our main result of this paper which shows that this algorithm enables us to transform a lossless bounded real linear complex quantum system in modal form into cascade from. This then defines a physical realization as a cascade of cavities.
**Theorem.** Consider an $m \times m$ lossless bounded real complex transfer function matrix $K(s)$ with a minimal modal canonical form quantum realization such that the eigenvalues of the matrix $\tilde{F}$ are all distinct and that all of the matrix inverses exist when the above algorithm is applied to the system. Then the vectors $H_1, H_2, \ldots, H_n$ defined in the above algorithm together with the eigenvalues $p_1, p_2, \ldots, p_n$ define an equivalent cascade quantum realization for the transfer function matrix $K(s)$. Furthermore, this system is such that the condition $p_i + p_i^* = -H_i^\dagger H_i$ is satisfied for all $i$. Moreover, the matrices $\{F, G, H, I\}$ defining this cascade quantum realization are related to the matrices $\{\tilde{F}, \tilde{G}, \tilde{H}, I\}$ defining the modal quantum realization according to the formulas:

$$\tilde{F} = T F T^{-1}, \quad \tilde{G} = T G, \quad \tilde{H} = H T^{-1}$$

where the matrix $T$ is defined in the above algorithm.
Illustrative Example

- To illustrate the main result of this paper, we consider a simple example.

- This quantum system is defined in terms of the coupling matrix $\Lambda$ and the Hamiltonian matrix $M$ which are as follows:

$$\Lambda = \begin{bmatrix} 2 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix}.$$ 

- From this, the matrices for an initial linear complex quantum system are constructed as follows:

$$F_1 = \begin{bmatrix} -2 - 10i & -1 \\ -1 & -0.5 - 4i \end{bmatrix}; \quad G_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix};$$

$$H_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}; \quad J_1 = 1.$$
The corresponding complex transfer function
\[ K(s) = H_1(sI - F_1)^{-1}G_1 + J_1 \]
is
\[ K(s) = \frac{s^2 - (2.5 - 14i)s - (40 + 13i)}{s^2 + (2.5 + 14i)s - (40 - 13i)}. \]

In order to apply our approach to find a cascade cavity quantum realization for this transfer function, we first transform the initial linear complex quantum system to modal form. This yields the following matrices:

\[ \tilde{F} = \begin{bmatrix} -2.0423 - 9.8399i & 0 \\ 0 & -0.4577 - 4.1601i \end{bmatrix}; \]
\[ \tilde{G} = \begin{bmatrix} -2.1225 + 0.1368i \\ -0.9458 - 0.3455i \end{bmatrix}; \]
\[ \tilde{H} = \begin{bmatrix} 2.0148 - 0.1579i & 0.9032 + 0.3158i \end{bmatrix}; \quad \tilde{J} = 1. \]
From this, we can observe that the matrix $\tilde{F}$ is Hurwitz and all columns of the matrix $\tilde{H}$ are non-zero.

We now apply our algorithm to this modal system. This yields a linear complex quantum system in cascade form defined as follows:

$$F = \begin{bmatrix} -2.0423 - 9.8399i & 0 \\ -1.8626 + 0.5196i & -0.4577 - 4.1601i \end{bmatrix};$$

$$G = \begin{bmatrix} -2.0168 + 0.1300i \\ -0.9032 + 0.3158i \end{bmatrix};$$

$$H = \begin{bmatrix} 2.0168 + 0.1300i & 0.9032 + 0.3158i \end{bmatrix}; \quad J = 1.$$
From this, we can then construct a cascade of two single-sided cavities to realize the complex transfer function.

To construct the two cavities, we first express the elements of the matrix $H$ in polar coordinates as follows:

$$H = \begin{bmatrix} 2.0210e^{0.0644i} & 0.9568e^{0.3364i} \end{bmatrix}.$$

From this, we calculate the required cavity coupling coefficients as

$$\kappa_1 = 2.0210^2 = 4.0845; \quad \kappa_2 = 0.9568^2 = 0.9155.$$

Then, the complex transfer function can be realized via a cascade of two cavities as follows:
In this cascade cavity realization, cavity 1 has a Hamiltonian operator $\mathcal{H}_1 = 9.8399a_1^\dagger a_1$ and cavity 2 has a Hamiltonian operator $\mathcal{H}_2 = 4.1601a_2^\dagger a_2$.

In this single-input single-output example, all of the phase-shifters can be combined together and cancel out. This leads to the simplified cascade realization shown below with the following matrices defining the corresponding linear complex quantum system in cascade form:

$$F_s = \begin{bmatrix} -2.0423 - 9.8399i & 0 \\ -1.9337 & -0.4577 - 4.1601i \end{bmatrix};$$

$$G_s = \begin{bmatrix} -2.0210 \\ -0.9568 \end{bmatrix};$$

$$H_s = \begin{bmatrix} 2.0210 & 0.9568 \end{bmatrix}; \quad J = 1.$$
\[ \Delta_1 = -9.8399 \]

\[ \kappa_1 = 4.0845 \]

\[ \Delta_2 = -4.1601 \]

\[ \kappa_2 = 0.9155 \]